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Chapter 1

SEMISIMPLE NORMAL INJECTIVE KRASNER HYPERMODULES

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1. INTRODUCTION

Algebraic hyperstructures put forth a natural generalization of classical algebraic structures, and in 1934 they were introduced by Marty (Marty, 1934:45-49) at the eighth Congress of Scandinavian mathematicians where he generalizes the concept of group to the concept of hypergroup. A hypergroup, a non-empty set equipped with relational hyperprocessing and reproductive hyperprocessing. In a group, the composition of two elements is an element, whereas in a hypergroup the composition of two elements is an element, a non-empty set. Since then, from a theoretical point of view, many different types of hyperstructures (hyperring, hypermodule, hypervector space,...) and applications of pure and applied mathematics to many subjects (Corsini,1993; Corsini & Fotea,2003; Davvaz, 2012; Davvaz & Fotea,2007; Vougiouklis,1994). There are different hyperrings in the literature. A special case of this species is the introduced hyperring by Krasner (Krasner,1983:307-312). Furthermore, Krasner introduced a new class of hyperrings and hyperfields: the quotient (factor) hyperrings and hyperfields.

The rest of this section is arranged as follows: we remind some basic knowledge of definitions related to the hyperstructures used throughout the article. We define semisimple normal injective Krasner hypermodules. We also investigate some properties of such hypermodules and prove that there is semisimple normal injective Krasner hypermodules have properties similar to those of strongly injective modules (Türkmen & Nişancı Türkmen, 2021:1-7).

2. BASIC DEFINITIONS IN THEORY OF HYPERMODULES

We give some fundamental definitions of hyperstructures used (Davvaz,2012; Davvaz & Fotea,2007) in this book chapter. Let X be a non-empty set. Then, a mapping $\circ: X \times X \longrightarrow P^*(X)$ is called a "*binary hyperoperation*" on X, where $P^*(X)$ is the family of all non-empty subsets of X. The couple $(X,\circ)$ is called "*hypergroupoid*". In this definition, if U and V are non-empty subsets of X and $x \in X$, then we define $U \circ V = \bigcup_{u \in U, v \in V} u \circ v$, $x \circ U = \{x\} \circ U$ and

$$U \circ x = U \circ \{x\}$$

A hypergroupoid $(X,\circ)$ is called a "*semihypergroup*", if for every $x, y, z \in X$, we have $x \circ (y \circ z) = (x \circ y) \circ z$; a "*quasihypergroup*" if for every $x \in X$,

$$x \circ X = X = X \circ x$$

a "*hypergroup*" if it is semihypergroup and a quasihypergroup; a "*commutative hypergroup*" if the hyperoperation $\circ$ is commutative on the set of X. A "*Krasner hyperring*" is an algebraic structure $(R, +, .)$ which satisfies the following conditions:

1. $(R, +)$ is a commutative hypergroup,
2. there is a $0 \in R$ such that $0 + x = \{x\}$ for every $x \in R$,
3. there is a unique $x' \in R$ such that $0 \in x + x'$ (x' is denoted by $-x$) for every $x \in R$,
4. $z \in x + y$ implies that $y \in -x + z$ and $x \in z - y$,
5. $(R, .)$ is a semigroup which has zero as a bilaterally absorbing element, i.e. $x.0 = 0.x = 0$.
6. the multiplication hyperoperation "." is distributive with respect to the hyperoperation "+".

By the definition, it is clearly seen that every ring is a Krasner hyperring. For the basic example of Krasner hyperring, we refer to the reader (Davvaz, 2013).

Let $(R, +, .)$ be a Krasner hyperring and S be a non-empty subset of R. Then S is called a "*subhyperring*" of R *if* $(s, +, .)$ is itself a hyperring. A subhyperring S of a Krasner hyperring $(R, +, .)$ is a "*hyperideal*" of R if

$$r.a \in S$$

and $a.r \in S$ for every $a \in S, r \in R$. A commutative Krasner hyperring $(R, +, .)$ with identity element "1" is *a "Krasner hyperfield"* if $(R \setminus \{0\}, .)$ is a group. Let $(M, +)$ be a hypergroup and $(R, +, .)$ be a hyperring. According to (Vougiouklis,1994) M is called a "*left hypermodule over R*" if there exists : $R \times M \longrightarrow P^*(M)$; $(a, m) \longmapsto a.m$ such that for every $a, b \in R$ and $m, m_1, m_2 \in M$, we have:

1. $a.(m_1 + m_2) = a.m_1 + a.m_2$,
2. $(a + b).m = (a.m) + (b.m)$,
3. $(a.b).m = a.(b.m)$.

If R is a Krasner hyperring and $(M, +)$ is a canonical hypergroup which satisfies the above conditions taking an external operation $\cdot: R \times M \longrightarrow M$ by $(r, m) \longmapsto r.m \in M$, and $r.0 = 0$, then M is called a "*left Krasner R-hypermodule*". A left Krasner hypermodule M over R is called "*unitary*" if $1_R.a = a$ for every $a \in M$. In terms of convenience, by "an R-hypermodule" we mean an unitary left Krasner R-hypermodule. A non-empty subset N of an R-hypermodule M is said an "*R-subhypermodule*" of M denoted by $N \leq M$ if N is an R-hypermodule itself. If $N \subset M$ and N is a subhypermodule of M, N is called a "*proper subhypermodule*" of M. It is easy to prove that a non-empty subset N of an R-hypermodule M if for every $x, y \in N$ and every $r \in R$, $x - y \subseteq N$ and $r.x \in N$ The subset $Ra = \{ ra \mid r \in R \} \leq N$ for every element a of an R-hypermodule N. If $N \leq M$, M is called an "*extension*" of N. Let M and N be R-hypermodules. A function $f: M \longrightarrow N$ that satisfies the conditions.

1. $f(m_1 + m_2) = f(m_1) + f(m_2)$,
2. $f(rm) = rf(m)$

for every $r \in R$, and every $m, m_1, m_2 \in M$, is said to be a "*strong R-homomorphism*" from M into N. If N is an R-hypermodule and $f: M \longrightarrow N$ is a strong R-homomorphism, $\ker(f) = \{ m \in M \mid f(m) = 0_N \}$. Moreover

$$Im(f) = \{ n \in N \mid \exists\, m \in M : \ n \in f(m) \}$$

Let M be a hypermodule over a hyperring R and N be a subhypermodule of M. Consider the set $\frac{M}{N} = \{m + N \mid m \in M\}$, then $\frac{M}{N}$ is a hypermodule over R

under hyperoperation is defined as $+: \frac{M}{N} \times \frac{M}{N} \longrightarrow P^*(\frac{M}{N})$ and $\cdot: R \times \frac{M}{N} \longrightarrow \frac{M}{N}$ such that $(m_1 + N) + (m_2 + N) = \{a + N \mid a \in m_1 + m_2\}$ and $r.(m + N) = r.m + N$ for every $m, m_1, m_2 \in M$ and $r \in R$. The set $\frac{M}{N}$ that satisfies the above conditions is called a "*quotient (factor) hypermodule*" according to N subhypermodule of M. Note that $m + N = N$ if and only if $m \in M$. In (Mahjoob & Ghaffari, 2018:554-568), a non-zero R-hypermodule M is called "*simple*", if the only subhypermodules of M are $\{0\}$ and M. We denote by $S(M)$, the set of all simple subhypermodules of an R-hypermodule M. Let M_1 and M_2 be subhypermodules of R-hypermodule M. Then M is called "*independent*", if $M_1 \cap M_2 = \{0\}$. If M_1 and M_2 are independent, then $M_1 + M_2$ is denoted by $M_1 \oplus M_2$. Also, a subhypermodule N of M is called a "*direct summand*" of M, if $M = N \oplus K$, for some subhypermodule K of M by (Talaee,2013:5-14). Let M be an R-hypermodule. M is called "*semisimple*", if for every subhypermodule K of M, there exists a subhypermodule N of M such that $M = K \oplus N$ (Mahjoob & Ghaffari, 2018: 554-568). Let M be a hypermodule and $\mathcal{X} = \{K_j \mid j \in J\}$ be a set of subhypermodules of M for any index set. The hypermodule M has "*the commutative property for sums (CPS) on $\mathcal{X}$*" if for every subset I of J, if $\sum_{i \in I} K_i = \sum_{\alpha \in \theta} K_\alpha$ where θ is a permutation of I. If M is a Krasner R-hypermodule then M satisfies the condition CPS on the set of its all subhypermodules in (Hamzekolaee et al.,2021:131-145) as the sum of all small subhypermodules of M, i.e. $Rad(M) = \sum_{L \ll M} L$. If M has no small subhypermodules of M, then $Rad(M) = M$.

Let M be a hypermodule. In (Hamzekolaee et al.,2021:131-145), a proper subhypermodule N of M is called "*small*" in M, denoted by $N \ll M$, if

$$M \neq N + L$$

for every proper subhypermodule L of M. For these subhypermodules U and V of M, V is called a "*supplement*" of U in M if $M = U + V$ and $U \cap V \ll V$ i.e. the canonical mapping $V \longrightarrow \frac{M}{U}$ is a small strong epimorphism. A subhypermodule U of M has "*ample supplements*" in M, if, whenever $U + V = M$, V contains a supplement V' of U in M. Since every direct summand is a supplement subhypermodule.

3. SOME RESULTS ON SEMISIMPLE NORMAL INJECTIVE KRASNER HYPERMODULES

In (Ameri & Shojaei, 2020), the basic features of injective modules have been transferred to hypermodules as normal injective. An R-hypermodule M is called "*normal injective*" if for every strong monomorphism $g \in Hom_R(A, B)$ and every strong homomorhism $f \in Hom_R(A, M)$, there exists $\bar{f} \in Hom_R(B, M)$ such that $\bar{f} \circ g = f$. In addition, the Baer criterion used in module theory is defined by moving to hypermodules as Baerian injective in (Ameri & Shojaei, 2020). Inspired by (Zöschinger,1974:267-287), we can easily define a hypermodule M with properties (E) and (EE). Through the rest of this section, we focus on the notion of semisimple normal injective hypermodules and the unitary left Krasner R-hypermodule over a Krasner hyperring R will be studied wherever the concept of Krasner hypermodule is written.

Definition 3.1 (a) Let M be a Krasner hypermodule. We call M "*semisimple normal injective*" if whenever $M + L = N$ with $M \leq N$, there exists a subhypermodule L' of L such that $M \oplus L' = N$. It is easily proven that every semisimple normal injective hypermodule is normal injective.

(b) Let M be a Krasner hypermodule. We call a hypermodule M has "*the property (E)*" if M has a supplement in every extension N as a proper generalization of normal injective hypermodules, and a hypermodule M has "*the property (EE)*" if M has ample supplements in every extension N.

Recall from (Bordbar et al., 2020:1-19) that a hypermodule M is called "*Artinian*" if it satisfies the descending chain condition on subhypermodules of M for every descending chain of subhypermodules $M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots$ there exists $N \in \mathbb{N}$ such that $M_n = M_N$, for every natural number $n \geq N$, this is equivalent condition with every descending chain of subhypermodules has a minimal element.

By the Definition 3.1(b), the following lemma is obtained clearly using (Zöschinger,1974: Lemma1.2).

Lemma 3.2 Let M be a Krasner hypermodule. Then every subhypermodule of M has the property (E) if and only if M has the property (EE).

Recall from (Ameri & Shojaei, 2020) that an R-hypermodule P is "*normal projective*" if for every strong epimorphism $g \in Hom_R(A, B)$ and every strong homomorphism $f \in Hom_R(P, B)$, there exists $\bar{f} \in Hom_R(P, A)$ such that

$$g \circ \bar{f} = f$$

If $f: P \longrightarrow M$ be a small strong epimorphism and P is projective R-hypermodule, P is called a "*projective cover*" of M.

Definition 3.3 Let R be a Krasner hyperring. R is called "*left perfect*" if every left Krasner R-hypermodule has a projective cover. If every simple R-hypermodule is normal injective, the Krasner hyperring R is called "*left V-hyperring*".

In the next Corollary, we will characterize properties (E) and (EE) in Krasner hypermodules in Krasner hyperrings.

Corollary 3.4 The following statements are equivalent for a Krasner hyperring R.

1. R is left perfect.
2. Every left Krasner R-hypermodule has the property (E).
3. Every left Krasner R-hypermodule has the property (EE).

Proof. Clear by Lemma 3.2 and (Clark et al., 2006). ■

In the following Example, we give an example for a Krasner hypermodule which has the property (E) but not normal injective.

Example 3.5 Let R be the Krasner hyperring $\frac{\mathbb{Z}}{n\mathbb{Z}}$ for $n > 1$ and the hypermodule $M = {}_R R$. Since R is an artinian hyperring, M has the property (E) by Corollary 3.4. But M is not normal injective.

The following Theorem is an answer to the question: "Whose hypermodule classifications of left Krasner V-hyperring?"

Theorem 3.6 Let R be a Krasner hyperring. Then the following statements are equivalent.

1. Every Krasner R-hypermodule with the property (EE) is semisimple normal injective.
2. Every Artinian Krasner R-hypermodule is semisimple normal injective.
3. R is a left Krasner V-hyperring.

Proof. (1)⟹(2) Clear as artinian hypermodules satisfy the property (EE).

(2)⟹(3) Let M be a simple Krasner R-hypermodule. By the hypothesis, M is (semisimple) normal injective. Thus R is a Krasner left V-hyperring.

(3)⟹(1) Let M be a hypermodule with the property (EE) and N be an extension of M. So there exists a subhypermodule L of N such that $N = M + L$. Since M has the property (EE), there exists a subhypermodule L' of L such that $N = M + L'$ and $M \cap L' \ll L'$. Thus, $M \cap L' \subseteq Rad(L') = \{0\}$. Therefore M is semisimple normal injective. ■

Now also the following property of semisimple normal injective Krasner hypermodules which is easily proven:

Lemma 3.7 The class of semisimple normal injective Krasner hypermodules is closed under strong isomorphism.

Proof. Let $f: M \longrightarrow K$ be a strong isomorphism and $K \leq N$. Suppose that M is a semisimple normal injective Krasner hypermodule. Consider the following diagram in Table 1:

Table 1

$$\begin{tikzcd} M \arrow[r, "\iota f"] & N \arrow[d, "I_N"] \\ K \arrow[u, "f^{-1}"] \arrow[r, "\iota"] & N \end{tikzcd}$$

where $\iota: K \longrightarrow N$ is the inclusion mapping. Since $\iota f: M \longrightarrow N$ is a strong monomorphism, M is a subhypermodule of N. Let $N = K + L$ for some subhypermodule of L$\leq N$. Then,

$$\begin{aligned} N = I_N(N) = I_N(K + L) = I_N(K) + I_N(L) = (I_N\iota)(K) + L \\ = (\iota f)\big(f^{-1}(K)\big) + L = (\iota f)(M) + L = (\iota f)(M) \oplus L' \end{aligned}$$

with $L' \leq L$ since M is semisimple normal injective. Therefore $N = K \oplus L'$. So K is semisimple normal injective. ■

We give this result as a consequence of Lemma 3.7.

Theorem 3.8 Let M be a Krasner hypermodule. Then the following statements are equivalent.

1. M is semisimple normal injective.
2. Every subhypermodule of M is normal injective.

Proof. (**1**) $\Longrightarrow$ (**2**) Let $U \leq M$ and N be any extension of U. Let $W = \frac{V}{S}$, where the subhypermodule $S = \{(i_1(u), i_2(u)) \mid u \in U\} \leq V$ for these inclusions strong homomorphisms $i_1 : U \longrightarrow M$ and $i_2 : U \longrightarrow N$ for the external direct product V of M and N. Then consider strong monomorphisms $f : M \longrightarrow W$, $f(m) = (m, 0) + S$ and $g : N \longrightarrow W$, $g(n) = (0, n) + S$ for every $m \in M$, $n \in N$. If $(m, n) + S \in W$, then $(m, n) + S \in W$, then

$$(m, n) + S = \big((m, 0) + S\big) + \big((0, n) + S\big) = f(m) + g(n)$$

So $W = Im(f) + Im(g)$. Since $fi_1 = gi_2$, we obtain the following pushout diagram in Table 2:

Table 2

$$\begin{tikzcd}
U \arrow[r, "i_2"] \arrow[d, "i_1"'] & N \arrow[d, "g"'] \\
M \arrow[r, "f"] & W
\end{tikzcd}$$

Here, M is strong isomorphic to $Im(f)$. By the assumption M is semisimple normal injective by Lemma 3.7. Therefore, we have the decomposition $W = Im(f) \oplus L$ for some subhypermodule L of $Im(g)$. Now $L = g^{-1}(W) = g^{-1}(Im(f) \oplus L) = U + g^{-1}(L)$. Let $x \in U \cap g^{-1}(L)$. Then $x = 0$, as g is a strong monomorphism. Thus $U \cap g^{-1}(L) = \{0\}$. Hence U is normal injective.

(2) $\Longrightarrow$ **(1)** For any extension N of M, let $N = M + T$ for some subhypermodule T of N. By the hypothesis, $M \cap T$ is normal injective and so there exists a subhypermodule T' of T such that $(M \cap T) \oplus T' = T$. We have

$$N = M + T = M + \left((M \cap T) + T'\right) = M + T'$$

Since $(M \cap T) \cap T' = \{0\}$ and $T' \leq T$, then $M \cap T' = \{0\}$. So $N = M \oplus T'$. Thus M is semisimple normal injective. ■

There exists a normal injective Krasner hypermodule which is not semisimple normal injective:

Example 3.9 Let $M = {}_{\mathbb{Z}}\mathbb{Q}$. Then M is a normal injective Krasner hypermodule but not semisimple normal injective.

Therefore we can give the following corollary:

Corollary 3.10 If M is a semisimple normal injective Krasner hypermodule, then $Rad(M) = \{0\}$.

For the following proposition we refer to (Wisbauer, 1991). The proof is included for completeness.

Proposition 3.11 Let R be a Krasner hyperring. Then the following statements are equivalent.

1. The left R-hypermodule R is a semisimple Krasner hypermodule.
2. Every Krasner R-hypermodule is normal injective.
3. Every Krasner R-hypermodule is semisimple normal injective.

Proof. (1)⟺(2) It is similarly proven by (Wisbauer,1991).

(2)⟹ (3) Let M be a Krasner R-hypermodule. By the hypothesis, every subhypermodule of M is normal injective. By Theorem 3.8, M is semisimple normal injective.

(3)⟹(2) Clear. ■

By using “Theorem 3.8”, we obtain the following main result:

Lemma 3.12 Let M be a simple Krasner hypermodule. M is normal injective if and only if it is semisimple normal injective.

In the light of Lemma 3.12 we have the following corollary:

Corollary 3.13 Let R be a Krasner hyperring. Then the following statements are equivalent.

1. *R is a left V-hyperring.*

2. Every simple Krasner R-hypermodule is semisimple normal injective.

We now prove a related result.

Theorem 3.14 The class of semisimple normal injective Krasner hypermodules is closed under subhypermodules and factor hypermodules.

Proof. Let M be a semisimple normal injective Krasner hypermodule and

$$U \leq V \leq M.$$

It follows from Theorem 3.8 that U is normal injective. Again applying Theorem 3.8, we have that V is semisimple normal injective. In addition, since every factor hypermodule $\frac{M}{V}$ of M is a direct summand of M, any factor hypermodule $\frac{M}{V}$ of M is semisimple normal injective. ∎

On the other hand, we have:

Theorem 3.15 Let $0 \longrightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \longrightarrow 0$ be a short exact sequence for Krasner hypermodules which consists of strong homomorphisms. Then the following statements are equivalent.

1. M is semisimple normal injective.

2. M_1 and M_2 are semisimple normal injective.

Proof. Since the following exact sequences for Krasner hypermodules

$$0 \longrightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \longrightarrow 0$$

and

$$0 \longrightarrow Im(f) \xrightarrow{i} M \xrightarrow{\pi} \frac{M}{Im(f)} \longrightarrow 0$$

where i is the inclusion mapping (strong monomorphism) and π is the canonical strong epimorphism. Take the hypermodule M_1 as a subhypermodule of M and M_2 is strong isomorphic to $\frac{M}{M_1}$ loss of generality.

$(\mathbf{1}) \Longrightarrow (\mathbf{2})$ By Theorem 3.14, obvious.

$(\mathbf{2}) \Longrightarrow (\mathbf{1})$ By using the hypothesis, we take that M_1 and $\frac{M}{M_1}$ are semisimple normal injective. For $M \leq N$, let $N = M + T$, $\frac{N}{M_1} = \frac{M}{M_1} + \frac{T+M_1}{M_1}$. Since $\frac{M}{M_1}$ is semisimple normal injective, $\frac{N}{M_1} = \frac{M}{M_1} \oplus \frac{T'}{M_1}$ for some subhypermodule $\frac{T'}{M_1} \leq \frac{T+M_1}{M_1}$. Therefore, $N = M + T'$ and $(M \cap T') \oplus U$ for some subhypermodule $U \leq T'$. It follows that $N = M + T' = M + (M \cap T') \oplus U = M \oplus U$ with $U \leq T'$. Therefore M is semisimple normal injective. ■

Now we close this section with the following elementary observation:

Corollary 3.17 If $M = M_1 + M_2 + \cdots + M_n$, where each M_i is a semisimple normal injective Krasner hypermodule, then M is semisimple normal injective.

Proof. The external product of Krasner hypermodules $M_1, M_2, \ldots, M_n$ say N. By Theorem 3.16, N is semisimple normal injective. Then it follows from Theorem 3.14 that M is semisimple normal injective. ■

Finally we have the following implications on subhypermodules:

Normal Injective Krasner Hypermodule⟹Semisimple Normal Injective Krasner Hypermodule

4. CONCLUSION

In this chapter, semisimple normal injective left Krasner R-hypermodules is defined on a Krasner hyperring R and the features provided by these hypermodules are associated with the concept known as normal injective hypermodule in literature. For this association, first of all, (E) and (EE) properties on hypermodules and the concepts of semisimple normal injective hypermodules were defined. Indeed, we characterize semisimple normal injective Krasner hypermodules via strongly injective modules. Studies on strongly injective modules can be accessed in detail from (Türkmen & Nişancı Türkmen, 2021).

Open Problem: Studying the subject of this study on commutative hyperdomains especially Dedekind hyperdomains will ensure that it is found indecomposable and reduced parts of semisimple normal injective hypermodules.

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Chapter 2

ON QUASINORMAL SUBGROUPS

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1. INTRODUCTION

In the theory of groups, normality of a subgroup is of paramount importance. Such subgroups play an important role in determining the structure of a group. For instance quotient groups are constructed through normal subgroups. For an arbitrary group G, we call a subgroup H of G normal if every conjugate of H with each element of G is equal to the subgroup H again. There are weaker versions of normality of a subgroup such as quasinormality. Quasinormal subgroups are first introduced by Ore in his paper 'Structures and Group Theory I' in 1937. A subgroup H of a group G is called quasinormal if it permutes with all subgroups of G, i.e. H is called quasinormal if $KH = HK$ for all subgroups K of G. Ore also showed that quasinormal subgroups are subnormal and modular. Many studies have been done after Ores' paper. Gross gave upper bounds for nilpotency class and derived length of a subgroup which is core-free and quasinormal in a p-group in 'p-Subgroups of Core-free Quasinormal Subgroups' and 'p-Subgroups of Core-free Quasinormal Subgroups II'. Then Stonehewer proved the existence of non-soluble group which is generated by two metabelian quasinormal subgroups and also showed that a group generated by soluble quasinormal subgroups is locally soluble in the paper named 'Permutable Subgroups of Some Finite p-Groups'. If we take a look to more recent works Stonehewer reduced the studies to the class of p-groups in his paper 'Quasinormal Subgroups of Finite p-Groups' in 2010. Cossey and Stonehewer studied the special case where the quasinormal subgroup is abelian in 'Abelian

Quasinormal Subgroups of Finite Groups'. Studies on quasinormal subgroups are not restricted with finite groups. In 2014 Leone presented her paper 'Quasinormal Subgroups of Infinite Groups'.

In this section we give an account of relation between quasinormal subgroups and Frattini subgroup of a finite group.

2. PRELIMINARIES

2.1 Quasinormality

In this section, we begin with definition of quasinormal subgroups. All groups are finite in the Chapter.

Definition 2.1.1 Let G be a group and H, K be subgroups of G. H and K are said to be "*permutable*" if

$$HK = KH = \langle H, K \rangle$$

If H is permutable with all subgroups of G then H is called "*quasinormal*" in G. [Ore, 1937].

One can easily notice that every normal subgroup is quasinormal but the converse does not hold in general.

Definition 2.1.2 A group G is "*permutably decomposed*" if $G = AB$, where A and B are permutable. In this case we say that A and B are "*permutably contained*" in G. [Ore, 1939].

Following three theorems are useful for one who searches normal subgroups in a group.

Theorem 2.1.3 Let G be a group and $G = AB$ for some subgroups A and B of G. Then,

$$C = \{a \in A | \ bab^{-1} \in A, \ \forall b \in B\}$$

is normal in G. [Ore, 1939].

Theorem 2.1.4 Let G be a group and $G = AB$ for some subgroups A and B of G. Let $B_0 \trianglelefteq B$ and $B_0 \leq A \cap B$. Then there exists a subgroup A_0 of A such that $B_0 \leq A_0$ and $A_0 \trianglelefteq G$. [Ore, 1939].

Theorem 2.1.5 Let G be a group and $G = AB$ for some subgroups A and B of G and B be abelian. If $A \cap B \neq e$ then for every subgroup of H of A, H is normal in G. [Ore, 1939].

As we mentioned in Introduction quasinormal subgroups are modular too. Now it would be appropriate to give definition of modular subgroups of a group.

Definition 2.1.6 Let G be a group and M be subgroup of G. For all $H, K \leq G$ with $H \leq K$ if,

$$\langle H, M \rangle \cap K = \langle H, M \cap K \rangle \qquad (2.1)$$

and for all $H, K \leq G$ with $M \leq K$ if,

$$\langle H, M \rangle \cap K = \langle H \cap K, M \rangle \qquad (2.2)$$

then M is called "*modular*" subgroup of G. [Stonehewer, 2010].

For finite p-groups quasinormality and modularity of subgroups coincide.

The definition of modularity depends on two conditions, i. e. (2.1) and (2.2). If we only use the first one, we can give the definition below.

Definition 2.1.7 Let G be a group and M is a subgroup of G. M is called "*semimodular*" if (2.1) holds. of G. [Stonehewer, 2010].

Semimodularity may be enough for a subgroup to be quasinormal in a special case. See the Proposition below.

Proposition 2.1.8 Let G be a finite p-group and M be a semimodular subgroup of G. Then M is quasinormal [Stonehewer, 2010].

Similarly another definition could be done for the second condition of modularity. It is;

Definition 2.1.9 Let G be a group and M is a subgroup of G. M is called "*weak modular*" if (2.2) holds [Stonehewer, 2010].

Of course there exists a proposition for weakly modularity too.

Proposition 2.1.10 Let G be a finite p-group and M be a weakly modular subgroup of G. Then M is quasinormal [Stonehewer, 2010].

For a useful characterization of semimodularity we will give a proposition but we first need to define the maps ϕ and ψ.

Definition 2.1.11 Let G be a group and X, Y be subgroups of G. Then

$$\phi_{X,Y}: [X \,/\, X \cap Y] \to [\langle X, Y\rangle \,/\, Y]$$

by

$$\phi(A) = \langle A, Y\rangle$$

and

$$\psi_{X,Y}: [\langle X, Y\rangle \,/\, Y] \to [X \,/\, X \cap Y]$$

by

$$\psi(B) = X \cap B$$

Here for given subgroups $A \leq B$, the lattice of subgroups between A and B is denoted by $[A/B]$ [Stonehewer,].

Proposition 2.1.12 Let G be a group and A be a subgroup of G. Then,

(i) A is semimodular in G if and only if for all $X \leq G$

$$\phi_{X,A}\psi_{X,A} = id_{[X/X\cap A]}$$

(ii) if A is semimodular in G,

$$\psi_{A,X}\phi_{A,X} = id_{[\langle X,A\rangle/Y]}$$

Here id_A is the identity map of some set A. [Stonehewer, 2010].

2.2 Frattini Subgroup

For futher results we recall the Frattini Subgroup of a group.

Definition 2.2.1 Let G be a group. The intersection of all maximal subgroups is called the "*Frattini Subgroup*" of G and denoted by $\Phi(G)$ [Robinson, 1996]

Following theorem of Ore gives the relation between normal subgroups and maximal subgroups in a soluble group.

Theorem 2.2.2 Let G be a soluble group and $N \trianglelefteq G$. Then all maximal subgroups containing N are conjugate. Conversely all conjugate maximal subgroups contain the same normal subgroup N [Ore, 1937].

The theorem below presents the characterizations of finite nilpotent groups.

Theorem 2.2.3 Let G be a finite group. Then the followings are equivalent:

(i) G is nilpotent.
(ii) Every subgroup of G is subnormal.
(iii) G satisfies the normalizer condition.
(iv) Every maximal subgroup of G is normal.
(v) G is the direct product of its Sylow subgroups [Robinson, 1996].

As a conclusion of previous theorem one can easily notice that every finite p-group is nilpotent so it satisfies all the characterizations of the theorem.

Finally we give the following theorem which forms the fundamental idea of the definition in the next section.

Theorem 2.2.4 Let G be a finite group and M be a maximal subgroup of G. Then

(i) Either $Z(G) \leq M$ or $G' \leq M$
(ii) $G' \cap Z(G) \leq \Phi(G)$
(iii) $Z(G) \nleq M \Rightarrow M \lhd G$ [Rose, 1978].

Readers are referred to [Rose, 1978] and [Robinson, 1996] for necessary background.

3. ON QUASINORMAL SUBGROUPS OF FINITE p-GROUPS

In this section we shall investigate some properties of new subgroups, first we begin with defining them.

Definition 3.1 Let G be a group. Then $\Phi_{Z(G)}$ and $\Phi_{G'}$ are defined as,

$$\Phi_{Z(G)} = \bigcap_{Z(G) \leq M_i} M_i$$

and

$$\Phi_{G'} = \bigcap_{G' \leq M_i} M_i$$

where M_i are maximal subgroups of G.

Here $Z(G)$ and G' are the centre and the commutator subgroup of G.

We know that these sets are nontrivial if $Z(G)$ or G' is nontrivial and of course they are subgroups of G.

It is well known that both $Z(G)$ and G' are normal in any group G. But when it comes to the normality (or quasinormality) of $\Phi_{Z(G)}$ and $\Phi_{G'}$, G must have some special conditions.

A strike corollary follows from the definition.

Corollary 3.2 Let G be a finite group. Then $\Phi_{Z(G)} \cap \Phi_{G'} = \Phi(G)$

Next lemma is crucial forte subgroups above to be normal (quasinormal).

Lemma 3.3 Let G be a soluble group and M be a maximal subgroup of G. If M contains both $Z(G)$ and G' then all maximal subgroups contain both $Z(G)$ and G'. Otherwise the class of maximal subgroups of G is union of two disjoint subset.

Proof. If G is abelian then $G' = 1$ and $Z(G) = G$, so the proof is done. Then assume G is not abelian. Since G is finite then all maximal subgroups of G

contains either $Z(G)$ or G' by Theorem 2.2.4. Denote the class of maximal subgroups of G by

$$Cl(G,\lessdot) = \{M \leq G | M \lessdot G\}$$

and let

$$Cl_{Z(G)}(G,\lessdot) = \{M \in Cl(G,\lessdot) |\ Z(G) \leq M\}$$

$$Cl_{G'}(G,\lessdot) = \{M \in Cl(G,\lessdot) |\ G' \leq M\}$$

here $M \lessdot G$ means that M is a maximal subgroup of G.

Obviously $Cl_{Z(G)}(G,\lessdot)$ and $Cl_{G'}(G,\lessdot)$ are subsets of $Cl(G,\lessdot)$. And again by Theorem 2.2.4

$$Cl_{Z(G)}(G,\lessdot) \cup Cl_{G'}(G,\lessdot) = Cl(G,\lessdot)$$

For the first part of the lemma let M contains both $Z(G)$ and G'. Then

$$M \in Cl_{Z(G)}(G,\lessdot)$$

and $M \in Cl_{G'}(G,\lessdot)$. Since $Z(G)$ and G' are normal subgroups then each

$$M^* \in Cl_{Z(G)}(G,\lessdot)$$

must contain G' by Theorem 2.2.2. Similarly each $M_* \in Cl_{G'}(G,\lessdot)$ must contain $Z(G)$. So both $Cl_{Z(G)}(G,\lessdot)$ and $Cl_{G'}(G,\lessdot)$ consist of maximal subgroups which contain both $Z(G)$ and G'. Then

$$Cl_{Z(G)}(G,\lessdot) = Cl_{G'}(G,\lessdot) = Cl(G,\lessdot)$$

For the second part of the lemma let $M \in Cl_{G'}(G,\lessdot)$ and $M \notin Cl_{Z(G)}(G,\lessdot)$. Since G is not abelian then $Cl_{Z(G)}(G,\lessdot) \neq \emptyset$ and let $M^* \in Cl_{Z(G)}(G,\lessdot)$. If M^* contains $Z(G)$ then M and M^* would be conjugate and they would contain the same normal subgroups by Theorem 2.2.2. It means that $M \in Cl_{Z(G)}(G,\lessdot)$

Which is a contradiction. So

$$Cl_{Z(G)}(G,\lessdot) \cap Cl_{G'}(G,\lessdot) = \emptyset$$

For any finite group $\Phi_{G'}$ is normal (and quasinormal) in G by Theorem 2.2.4 (iii). But for $\Phi_{Z(G)}$ we have to restrict finite groups to finite p-groups. Obviously if G is a finite p-group, then G is nilpotent and every maximal subgroup of G is normal, so $\Phi_{Z(G)}$ is normal (and quasinormal) in G.

For simplicity we enumerate the conditions in Lemma 3.3 by (1) and (2), where

(1) $Cl_{Z(G)}(G,\lessdot) = Cl_{G'}(G,\lessdot)$

(2) $Cl_{Z(G)}(G,\lessdot) \cap Cl_{G'}(G,\lessdot) = \emptyset$

It is easy to see that for finite p-groups there is no other condition except these conditions by Lemma 3.3. ∎

Theorem 3.4 Let G be a finite p-group. Then $\Phi_{Z(G)}$is a unique minimal (normal) subgroup of G.

Proof. Since G is finite p-group $\Phi_{Z(G)}$ is normal (and quasinormal) in G by the explanations above.

Now consider the maps ϕ and ψ defined in Definition 2.1.11. Assume for every $X \leq G$

$$X \cap \Phi_{Z(G)} \leq H \leq X$$

then $\phi(H) = \langle H, \Phi_{Z(G)} \rangle$ and for $\Phi_{Z(G)} \leq \langle H, \Phi_{Z(G)} \rangle \leq \langle X, \Phi_{Z(G)} \rangle$

$$\psi\big(\langle H, \Phi_{Z(G)} \rangle\big) = \langle H, \Phi_{Z(G)} \rangle \cap X.$$

Since $\Phi_{Z(G)}$ is quasinormal then it is modular (and semimodular) by [Ore, 1937]. So,

$$\psi\big(\langle H, \Phi_{Z(G)} \rangle\big) = \langle H, \Phi_{Z(G)} \rangle \cap X = H$$

by Proposition 2.1.12 (i). Therefore $H = \langle H, \Phi_{Z(G)} \rangle$ because $H \leq X$. This implies that $\Phi_{Z(G)} \leq H$. Finally $\Phi_{Z(G)}$ is a unique minimal normal subgroup since H and X are arbitrary subgroups of G. ∎

Corollary 3.5 Let G be a finite p-group. Then $\Phi_{Z(G)} = \Phi_{G'} = \Phi(G)$.

Proof. The same way to prove Theorem 3.4 could be done for $\Phi_{G'}$ and since they are unique then $\Phi_{Z(G)} = \Phi_{G'}$. Also we have $\Phi_{Z(G)} \cap \Phi_{G'} = \Phi(G)$ by Corollary 3.2. So $\Phi_{Z(G)} = \Phi_{G'} = \Phi(G)$.

One may say that the previous corollary implies that only condition (1) holds for G. But we have to keep in mind that in condition (2) the intersection of maximals in both $Cl_{Z(G)}(G, \lessdot)$ and $Cl_{G'}(G, \lessdot)$ could be the same subgroup too.

We finish this section with applications of $\Phi_{Z(G)}$ and $\Phi_{G'}$ to Theorem 2.1.3, Theorem 2.1.4, Theorem 2.1.5. ■

Corollary 3.6 Let G be a group. Then,

$$C = \left\{a \in \Phi_{Z(G)} \middle|\ bab^{-1} \in \Phi_{Z(G)},\ \ \forall b \in \Phi_{G'}\right\}$$

is normal in $\langle \Phi_{Z(G)}, \Phi_{G'} \rangle$.

Proof. It is clear by Theorem 2.1.3 since $\Phi_{G'}$ is normal in $\langle \Phi_{Z(G)}, \Phi_{G'} \rangle$. ■

Corollary 3.7 Let G be a group, $B \trianglelefteq \Phi_{G'}$ and $B \leq \Phi(G)$. Then for some $B \leq A$, A is normal in $\langle \Phi_{Z(G)}, \Phi_{G'} \rangle$.

Proof. Clear by Theorem 2.1.4. ■

Corollary 3.8 Let G be a group, $\Phi_{G'}$ be abelian and $\Phi(G) \neq 1$. Then for some $H \leq \Phi_{Z(G)}$, H is normal in $\langle \Phi_{Z(G)}, \Phi_{G'} \rangle$.

Proof. Directly seen by Theorem 2.1.5. ■

4. CONCLUSION

The minimality of $\Phi_{Z(G)}$ and $\Phi_{G'}$ given in Theorem 3.4 could be combined with last three corollaries above. Readers may investigate more about the structure of $\Phi_{Z(G)}$ and $\Phi_{G'}$. These subgroups are located between $\Phi(G)$ and G. So there are at least two different normal series of G. Readers also may search those series. It seems there are a lot of fields to study on $\Phi_{Z(G)}$ and $\Phi_{G'}$.

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Chapter 3

A HYPERSTRUCTURAL APPROACH TO SS-SUPPLEMENTS

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1. INTRODUCTION

SS-supplemented modules which are studied in (Kaynar et al., 2020) as a strongly notion of an important class of supplemented modules. Supplemented modules have been extensively studied in recent years. Important features and characterizations of module theory can be found of (Clark et al., 2006), (Mohamed & Müller, 1990); Wisbauer, 1991). Meanwhile, none as the researchers so far have taken an ss-supplemented hyperstructural approach, while only it has been studied as a concept that has entered the literature in module theory. We will introduce this concept with the help of hypermodule, taking inspiration from ss-supplemented modules and studies on basic topics. Let M be a module. M is called "*ss-supplemented*" if every submodule N of M has a supplement K in M such that $N \cap K$ is semisimple in (Kaynar et al., 2020). As we know from definitions, a hypermodule may not contain an element like 0. Morever, the intersection of each two subhypermodules of an R- hypermodule is not a subhypermodule of that hypermodule, in general, So, we will work with Krasner hypermodule on Krasner hyperring, not working with the hypermodules on any hyperring. We refer the researchers for more details about hyperstructures theory to (Ameri&Shojaei,2020; Corsini,1994; Corsini& Leoreanu,2003; Davvaz,2012; Davvaz,2007; Hamzekolaee et al.,2021; Marty,1934; Talaee, 2013)

The remaining sections inside will be as follows, in section 2, basic definition and properties that will be read related to hyperstructures will be given, in section 3, the concept strongly local hypermodules and ss-supplemented hypermodules will be characterized by the help of hyperring by giving algebraic propertied by these concepts.

2. BASIC DEFINITIONS

Let H be a non-empty set and mapping $\circ: H \times H \longrightarrow P^*(H)$ where $P^*(H)$ is the set of every non-empty subset of H. Then the mapping " $\circ$ " is called a "*hyperoperation*" on H and the algebraic hyperstructures is based on this hyperoperation. Theory of hyperstructures was first introduced by Marty in (Mahjoob & Ghaffari, 2018). Many important developments have been presented by this concept and interest in this theory have been rised by algebraist till now. To prove this assertion, we refer readers (Corsini & Leoreanu, 2003; Hamzekolaee et al., 2021; Marty, 1934; Talaee, 2013), Talaee introduced and studied classical algebraic properties of small subhypermodules in hypermodules in the same way as the concept as in module theory. We specialize here this study to a more special contex.

In what follows, we give some basic definitions about hypergroups, hyperring and hypermodule which we need in this paper.

Let "$\circ$" be a hyperoperation on H. Then $(H,\circ)$ is called *a* "*hypergroupoid*". It is defined sets

$$X \circ Y = \bigcup_{\substack{x \epsilon X \\ y \epsilon Y}} x \circ y$$

and $X \circ \{a\} = x \circ a$ for $a \in H$ and $X, Y \in P^*(H)$. A hypergroupoid $(H,\circ)$ is called a "*semihypergroup*" if for every $a, b, c \in H$, we have $(a \circ b) \circ c = a \circ (b \circ c)$. A semihypergroup $(H,\circ)$ is called a "*hypergroup*" if

$$a \circ H = H \circ a = H$$

for every $a \in H$. A non-empty subset F of a hypergroup $(H,\circ)$ is called a "*subhypergroup*" if

$$f \circ F = F \circ f = F$$

for every $f \in F$. A hypergroup H is called "*commutative*" if the hyperoperation "∘" is commutative on the set of H. A commutative hypergroup $(H,\circ)$ is called "*canonical*", if the following three condition satisfies.

1. There exists a unique of H such that $a \circ 0 = \{a\}$ for every $a \in H$.
2. There exists a unique $a^{-1} \in H$ such that $a \in a \circ a^{-1}$ for every $a \in H$.
3. If $a \in b \circ c$, then $b \in a \circ c^{-1}$ and $c \in b^{-1} \circ a$ for every $a, b, c \in H$ (Davvaz, 2007).

The triple $(R,\uplus,\circ)$ is a hyperring, if $(R,\uplus)$ is a hypergroup, $(R,\circ)$ is a "*semihypergroup*" and "∘" is distributive over "⊎" (Davvaz, 2007). A hyperring $(R,\uplus,\circ)$ is called *Krasner*, if $(R,\uplus)$ is a canonical hypergroup and $(R,\circ)$ is a semigroup such that 0 is a zero element, i.e. $x \circ 0 = 0 = 0 \circ x$ for every $x \in R$ (Davvaz, 2007). Let $(R,\uplus,\circ)$ be a hyperring, $(M, +)$ a hypergroup and $*$ $: R \times H \longrightarrow f^*(EK2)$ an external hyperoperation $(H, +, .)$ is called *a* "*left R-hypermodule*" if it satisfies following statements for every $r_1, r_2 \in R, h_1, h_2 \in H$

1. $r_1(\,h_1 + h_2) = (r_1.\,h_1) + (r_2.\,h_2)$;
2. $(r_1 \uplus r_2).\,h_1 = (r_1 h_1) + (r_2 h_2)$
3. $(r_1 r_2).\,h_1 = r_1.\,(s_1 h_1)$ (Davvaz, 2007).

In similar way, as right hypermodule over R can be defined. If $(H, +)$ is a canonical hypergroup and $(R,\uplus,.)$ is a Krasner hyperring, then H is called "*canonical R-hypermodule"* where "." is an external operation, that is $.: R \times H \longrightarrow H$ by $(r, b) \longrightarrow r.\,b$ and $r.\,0 = 0$. A non-empty subset F of on (Krasner) R-hypermodule H is called *a* "*subhypermodule*", denoted by $F \leq H$, if F itself is a (Krasner) hypermodule over R with hyperoperation defined on $R \times H$. Let M be a Krasner R-hypermodule. A subhypermodule F is "*small*" in H (denoted by $F << H$), if $F + L = H$ implies $L = H$, where $L \leq H$. Equivalently, if L is a proper subhypermodule of H, then $F + L \neq H$ (Wisbauer, 1991). Let H be a Krasner hypermodule. H is called "*hollow*" if every proper subhypermodule of H is small in H and H is called "*local*" if H has a proper subhypermodule that contains all proper subhypermodules of H (Hamzekolaee et al., 2021). It is clear that every local hypermodule is hollow. Let H be a hypermodule over a hyperring. R and $F \leq H$. Consider set $H/F =$

$\{h + F | h \in H\}$. Then H/F is a hypermodule over R under hyperoperation defined by $+: H/F \times H/F \longrightarrow P^*(H/F)$ and external composition $.: R \times H/F \longrightarrow H/F$ such that

$$h_1 + F + h_2 + F = \{x + F | x \in h_1 + h_2\}$$

and $r.(h + F) = rh + F$ for every $h, h_1, h_2 \in H$ and $r \in R$. Let H be a hypermodule, J on indexed set and

$$\chi = \{F_j | j \in J\}$$

be a set of subhypermodules of H. It is called that H satisfies "*CPS (commutative property for sums) on* χ" if for every subset K of J. We have $\sum_{k\in K} F_k = \sum_{\gamma\in\Gamma} F_\gamma$, where Γ is a permutation of K. A Krasner R- hypermodule always satisfies CPS on the set of its all subhypermodules. Let H be a Krasner R-hypermodule such that H satisfies CPS on the set of its all small subhypermodules. The sum of all small subhypermodule of H defined by

$$Rad(H) = \sum_{F \ll H} F$$

If H has no small subhypermodule, $Rad(H) = H$ is assumed. Let H be a Krasner R- hypermodule. Then H is local if and only if it is local and $Rad(H) \neq H$ (Hamzekolaee et al., 2021). Let $(H_1, +_1, ._1)$ and $(H_2, +_2, ._2)$ be Krasner hypermodules over R. A mapping $f: H_1 \longrightarrow H_2$ is called a "*strong homomorphism*", if $f(x +_1 y) = f(x) +_2 f(y)$ and $f(r ._1 x) = r ._2 x$ for every $x, y \in H_1$ and $r \in R$ (Hamzekolaee et al., 2021). Let $f: H_1 \longrightarrow H_2$ be a strong homomorphism of Krasner hypermodules. The set of $Kerf = \{x \in H_1 | f(x) = 0_{H_2}\}$ is a subhypermodule of H_1 (Hamzekolaee et al., 2021). Let H be a Krasner R-hypermodule. A subhypermodule H is called a "*direct summand*" of H, if there exists a subhypermodule F of H such that $F \cap H = \{0\}$ and $F + L = H$ (Talaee, 2013). A non-zero R-hypermodule H is called *simple*, if the only subhypermodules of H are $\{0_H\}$ and H by (Mahjoob & Ghaffari, 2018). The set of all simple subhypermodules of an R-hypermodule H is denoted by $S(M)$. Let H be on R-hypermodule. Then H is called "*semisimple*" if for every subhypermodule F of H, there is a subhypermodule K of H such that $H = F \oplus K$ in (Mahjoob & Ghaffari, 2018). Let H be an R-hypermodule. H is called *supplemented* if for every subhypermodule F of H has a supplement in H, i.e. there exists a subhypermodule L of H such that $H = F + L$ and $F \cap L \ll L$ (Hamzekolaee et al., 2021). Let H be an R-hypermodule and F be a subhypermodule of H. If for every subhypermodule L of H such that $H = F + L$, there exists a supplement L' of F with $L' \leq L$, then we say F has "*ample supplements*" in H. If every subhypermodule of H has ample supplements in H, then H is called an "*amply supplemented hypermodule*" (Hamzekolaee et al., 2021).

3. STRONGLY LOCAL KRASNER HYPERMODULES

In this part, we define the notion of strongly local hypermodules and provide various properties of these hypermodules.

We call a Krasner hypermodule H "*strongly local*" if it is local and $Rad(H)$ is semisimple. The concept of strongly local hypermodule is a generalization of concept of simple hypermodules, but it is a specialized by the concept of local hypermodules. By $Soc(H)$, we denote by sum of all semisimple subhypermodule of H. Using by the radical of hypermodule H, i.e. $Rad(H)$, we specialized $Soc(H)$ to $Soc_S(H)$. We define a subhypermodule $Soc_S(H)$ as $\sum\{F \ll L | F \text{ is simple}\}$. We clearly seen that $Soc_S(H) \subseteq Soc(H)$ and $Soc_S(H) \subseteq Rad(H)$.

Let's start the section with the lemmas that we will use frequently.

Lemma 3.1 Let H be a Krasner hypermodule and F be a semisimple subhypermodule in $Rad(H)$. Then $F << H$.

Proof. Consider the subhypermodule L of H such that $H = F + L$. By the hypothesis, there exists a subhypermodule F' of F such that $F = (F \cap L) \oplus F'$. Thus $H = F + L = [(F \cap L) \oplus F'] + L = F' + L$. Since $F' \cap L = (F' \cap F) \cap L = F' \cap (F \cap L) = \{0_H\}$ by Lemma 2.11 of (Talaee, 2013), then $H = F' \oplus L$. It follows from $Rad(F') \subseteq Rad(F) = 0$ that $Rad(H) = Rad(L)$. So $H = F + L \subseteq Rad(H) + L \subseteq L$. Thus $F \ll H$. ■

Lemma 3.2 Let H be a Krasner hypermodule. Then $Soc_S(H) = Rad(H) \cap Soc(H)$.

Proof. It is clear that $Soc_S(H) \subseteq Rad(H) \cap Soc(H)$, for the converse inclusion, we take

$$x \in Rad(H) \cap Soc(H)$$

Then there exist $m \in \mathbb{N}^*$ and simple subhypermodules F_j of H $(1 \leq j \leq m)$ such that H is direct sums of every F_j. Since $Rx \ll H$, $F_j \ll H$ for all j. There are $x \in Rx \subseteq Soc_S(H)$. ■

Lemma 3.3 Let H be a Krasner hypermodule and F, L be subhypermodules of H. Then the following statements are equivalent.

1. $H = F + L$ and $F \cap L \subseteq Soc_S(L)$,
2. $H = F + L$, $F \cap L \subseteq Rad(L)$ and $F \cap L$ is semisimple,
3. $H = F + L$, $F + L \ll L$ and $F \cap L$ is semisimple.

Proof. (1) ⟹ (2) It follows from $F \cap L \subseteq Soc(L) \cap Rad(L)$ that $F \cap L \subseteq Rad(L)$ and $F \cap L$ is semisimple.

(2) ⟹ (3) Clear by Lemma 3.1.

(3) ⟹ (1) Clear by Lemma 3.2. ■

We say a non-zero Krasner hypermodule H "*indecomposable*" if the only direct summands of H are $\{0_H\}$ and H.

Lemma 3.4 Let H be a Krasner hypermodule, Then, H is simple or $Soc(H) \subseteq Rad(H)$.

Proof. Suppose that H is not simple. Let $H = Soc(H) + L$ for some subhypermodule L of H. Since $Soc(H)$ is semisimple, there exists a subhypermodule F of $Soc(H)$ such that $Soc(H) = (Soc(H) \cap L) \oplus F$. Thus,

$$H = Soc(H) + L = [Soc(H) \cap L \oplus F] + L = L \oplus F$$

Since F is indecomposable but not simple, then $F = H$. Since $Soc(H) \ll H$, $Soc(H) \subseteq Rad(H)$. ■

Corollary 3.5 Let H be a local Krasner hypermodule which is not simple. Then $Soc_S(H) = Soc(H)$.

We call a Krasner hypermodule H "*radical*" if $Rad(H) = H$. We denote by $P(H)$, the sum of all radical subhypermodules of H. Here $P(H)$ is the largest radical subhypermodule of H. If $P(H) = 0$, H is called "*reduced*".

Let us give a proposition to help classify strongly local Krasner hyperring.

Proposition 3.6 Let H be a strongly local Krasner hypermodule. Then H is reduced.

Proof. By the hypothesis, $P(H) \subseteq Rad(H) \subseteq Soc(H)$. So $P(H)$ is semisimple. Therefore

$$P(H) = Rad\big(P(H)\big) = 0$$

In the following proposition the main feature of strongly local Krasner hypermodule. ■

Proposition 3.7 Every factor hypermodule of a strongly local Krasner hypermodule is strongly local.

Proof. Let H be a strongly local Krasner hypermodule and F be a subhypermodule of H. We have H/F is a local hypermodule. Consider the strongly epimorphism $p: H \longrightarrow H/F$. Since

$$Rad(H/F) = Rad(H)/F \subseteq p(Soc(H)) \subseteq Soc(H/F)$$

H/F is strongly local. ■

4. SS-SUPPLEMENTED KRASNER HYPERMODULES

In this part, we define notion of ss-supplemented Krasner hypermodules and we study this notion comparatively with the notion of strongly local hypermodules.

Let H be a Krasner R-hypermodule H is called "*ss-suplemented*" if every subhypermodule F of H has a supplement L of H such that $F \cap L$ is semisimple. Let H be a Krasner R-hypermodule and F be a subhypermodule of H. If for every subhypermodule L of H such that $H = F + L$, there exists a ss-supplement L' of F with $L' \leq L$, then we call F has "*ample ss-supplements*" in H. If every subhypermodule of H has ample ss-supplements in H, then H is called an "*ample ss-supplemented*" hypermodule.

The relationship of strongly local hypermodule with maximal subhypermodules is given in the next proposition.

Proposition 4.1 Let H be a Krasner hypermodule and F be a maximal subhypermodule of H. A subhypermodule L of H is an ss-supplement of F in H if and only if $H = F + L$ and L is strongly local.

Proof. (⟹) Let L be an ss-supplement of F in H. Since $F \cap L$ is semisimple, then L is local, $F \cap L = Rad(L)$ is the unique maximal subhypermodule of L and $Rad(L) \subseteq Soc(L)$. So, L is strongly local.

(⟸) Since L is local and $H = F + L$, we have $F \cap L \subseteq Rad(L)$. By the hypothesis, $F \cap L$ is semisimple. Thus, L is an ss-supplement of F in H. ■

Lemma 4.2 Let H be an ss-supplemented Krasner hypermodule and $F \ll H$. Then $F \subseteq Soc_s(H)$.

Proof. By the hypothesis, H is the unique ss-supplement of F in H. Therefore $F \cap H = F$ is semisimple. By using Lemma 3.2, $F \subseteq Soc_s(H)$. ■

Corollary 4.3 Let H be an ss-supplemented Krasner hypermodule and $Rad(H) \ll H$. Then $Rad(H) \subseteq Soc(H)$

Proposition 4.4 Every strongly local Krasner hypermodule is amply ss-supplemented.

Proof. Let H be a strongly local Krasner hypermodule. Then H is local. So, H is amply supplemented. Since H has no supplement subhypermodule not including $\{0_H\}$ and H. It follows from $Rad(H) \subseteq Soc(H)$ that H is amply ss-supplemented. ■

Proposition 4.5 Let H be a hollow Krasner hypermodule. Then H is (amply) ss-supplemented if and only if it is strongly local.

Proof. (⟸) Clear by Proposition 4.4

(⟹) Suppose that H is ss-supplemented. Let $x \in Rad(H)$. Then $Rx \ll H$. So, by Lemma 4.2, $Rx \subseteq Soc_s(H)$.

We have $x \in Soc(H)$ and $Rad(H) \subseteq Soc(H)$. Suppose that H is radical. Then we have $H = Soc(H)$ and $Rad(H) = 0 = H$. This contradicts that H is hollow. Therefore $H \neq Rad(H)$ and so H is strongly local. ■

Example 4.6 Consider the $\mathbb{Z}$-hypermodule $H = \mathbb{Z}_{p^\infty}$ for the hyperring of integer $\mathbb{Z}$ and any prime integer p. Since hypermodule H is hollow, H is amply supplemented, but not amply ss-supplemented by Proposition 4.5.

Let us show under what conditions the ss-supplemented Krasner hypermodule with supplemented hypermodules will equivalent conditions.

Lemma 4.7 Let H be a supplemented Krasner hypermodule and $Rad(H) \subseteq Soc(H)$. Then H is ss-supplemented.

Proof. Let F be a subhypermodule of H. By the hypothesis, there exist a subhypermodule L of H such that

$$H = F + L$$

and $F \cap L \ll L$. Then $F \cap L \subseteq Rad(L) \subseteq Rad(H)$. Since $Rad(H) \subseteq Soc(H)$, then $F \cap L$ is semisimple. Therefore, L is a ss-supplement of the subhypermodule F in H. So, H is ss-supplemented. ■

Theorem 4.8 Let H be a Krasner hypermodule with small radical. Then the following statements are equivalent.

1. H is ss-supplemented
2. H is supplemented and $Rad(H)$ has an ss-supplement in H.
3. H is supplemented and $Rad(H) \subseteq Soc(H)$

Proof. (1)⟹(2) Clear

(2)⟹(3) By Lemma 4.2

(3)⟹(1) Follows from Lemma 4.7 ■

Lemma 4.9 Let H be a Krasner hypermodule and F, L be subhypermodules of H with F ss-supplemented. If $F + L$ has an ss-supplement in H, L also has an ss-supplement in H.

Proof. Let H be an ss-supplement of $F + L$ in H and V is an ss-supplement of $(U + L) \cap F$ in F. Then we have $H = U + V + L$ and $(U + V) \cap L \ll U + V$ by Corollary 2.5 of (Talaee, 2013). In addition, $U \cap (V + L)$ is a semisimple subhypermodule of H. Since $V \cap [(U + L) \cap F] = V \cap (U + L)$ is semisimple, $(U + V) \cap L$ is semisimple. Hence $U + V$ is an ss-supplement of L in H. ■

Proposition 4.10 Let F, L be any subhypermodules of a Krasner hypermodule H with $H = F + L$. Then if F and L are ss-supplemented, H is ss-supplemented.

Proof. Let T be any subhypermodule of H. $\{0_H\}$ is a ss-supplement of $H = F + L + T$ in H. Since F is ss-supplement, $L + T$ has an ss-supplement in H by Lemma 4.9, Again applying Lemma 4.9, we also obtain that T has an ss-supplement in H. So, H is ss-supplemented. ■

We have seen that the amply ss-supplemented Krasner hypermodule feature is completely inherited in factor hypermodules.

Proposition 4.11 If H is a (amply) ss-supplemented Krasner hypermodule, then every factor hypermodule of H is (amply) ss-supplemented.

Proof. Let H be an ss-supplemented Krasner hypermodule and H/F' be a factor hypermodule of H. By hypothesis, there exists a subhypermodule F of H with contains F' such that $H = F + L$, $F \cap L \ll L$ and $F \cap L$ is semisimple. Let $p: H \longrightarrow H/F'$ be a strong epimorphism. Then we have $H/F' = F/F' + (L + F')/F'$ and

$$F/F' \cap (L + F')/F' = \big((F \cap L) + F'\big)/F' = p(F \cap L) \ll p(L)$$
$$= (L + F')/F'$$

by Proposition 2.6 of (Talaee, 2013). Since $F \cap L$ is semisimple, $p(F \cap L) = F/F' \cap (L + F')/F'$ is semisimple. Thus, $(L + F')/F'$ is an ss-supplement of F/F' in H/F'. It can be similarly proven that if H is amply ss-supplemented, then so is every factor hypermodule of H. ■

Theorem 4.12 Let $H = \oplus_{j\in J} H_j$, where H_j is a strongly local Krasner hypermodule. Then H is ss-supplemented.

Proof. Since every strongly local hypermodule is local, $Rad(H_j) \subseteq Soc(H_j)$ for every $j \in J$. So

$$Rad(H) = \oplus_{j\in J} Rad(H_j) \subseteq \oplus_{j\in J} Soc(H_j) = Soc(H)$$

By Lemma 3.1, $Rad(H) \ll H$. It follows from Theorem 4.8 that H is ss-supplemented.

Recall that a Krasner hypermodule F is called "*H-generated*" for a Krasner hypermodule H if there exists a strongly epimorphism $\alpha: H^{(J)} \longrightarrow F$ for some index set J. ■

Corollary 4.13 Let H be a strongly local Krasner hypermodule. Then every H-generated hypermodule is ss-supplemented.

Proof. Suppose that F is H-generated. Then there exists a strong epimorphism $\alpha: F^{(J)} \longrightarrow H$ for some index set J. By Theorem 4.12, $F^{(J)}$ is ss-supplemented. Therefore, H is ss-supplemented by Proposition 4.11. ■

Proposition 4.14 Let H be a Krasner hypermodule. If every subhypermodule of H is ss-supplemented, then H is amply ss-supplemented.

Proof. Let F and L be subhypermodules of H such that $H = F + L$. Since L is ss-supplemented, there exists a subhypermodule L' of L such that $L = (F \cap L) + L'$, $F \cap L' \ll L'$ and $F \cap L'$ is semisimple. Then we have

$$H = F + L = F + \left((F \cap L) + L'\right) = F + L'$$

So, F has ample ss-supplements in H. Therefore, H is amply ss-supplemented. ■

The relationship between ss-supplemented subhypermodules and amply ss-supplemented Krasner hypermodules will be presented in the following lemma.

Lemma 4.15 Let H be amply ss-supplemented Krasner hypermodule and L be an ss-supplement subhypermodule in H. Then L is amply ss-supplemented.

Proof. Let L be an ss-supplement of a subhypermodule F of H. Let U and V be subhypermodules of L such that $L = U + V$. Then $H = (F + U) + V$. Since H is amply ss-supplemented, $F + U$ has an ss-supplement $V' \subseteq V$ in H. It follows from $U + V' \leq L$ that $L = U + V'$ So $L = U + V'$. Moreover $U \cap V' \ll V'$, as

$$U \cap V' \subseteq (F + U) \cap V' \ll V'$$

Since $(F + U) \cap V'$ is semisimple, $U \cap V'$ is semisimple. So V' is an ss-supplement of U in L. Therefore, L is amply ss-supplemented. ■

Theorem 4.16 Let H be a Krasner hypermodule. Then H is amply ss-supplemented if and only if every subhypermodule F of H is of the form $F = U + V$, where U is ss-supplemented and $V \subseteq Soc_s(F)$.

Proof. (⟹) Let F be a subhypermodule of H. Since H is ss-supplemented, F has an ss-supplement T in H. Say $V = F \cap T$. Since T is an ss-supplement of F in H, we obtain that $V \subseteq Soc_s(T) \subseteq Soc_s(F)$. Applying Lemma 2.11 of (Talaee, 2013), we have $F = F \cap F = F \cap (U + T) = U + F \cap T = U + V$. By Lemma 4.15, U is ss-supplemented.

(⟸) Let F be a subhypermodule of H. By the hypothesis, there exist subhypermodules U and V of H such that $F = U + V$, U is ss-supplemented and $V \subseteq Soc_s(F)$. By Proposition 4.10, F is ss-supplemented. Hence F is amply ss-supplemented by Proposition 4.14. ■

Corollary 4.17 The following statements are equivalent for a Krasner hypermodule H.

1. H is amply ss-supplemented,
2. Every subhypermodule of H is ss-supplemented,
3. Every subhypermodule of H is amply ss-supplemented.

We call a Krasner hypermodule H "*normal π-projective*" if whenever F and L subhypermodules of H such that $H = F + L$, there exists a strong endomorphism α of H such that $\alpha(H) \leq F$ and $(1 - \alpha)(H) \leq V$. Using the definition normal projective hypermodule in (Ameri & Shojaei, 2020), it is obtained that every normal projective module is normal π-projective.

We will show that the concepts of ss-supplemented Krasner hypermodules and amply ss-supplemented Krasner hypermodules have the same structure thanks to notion of normal π-projectively.

Proposition 4.18 Let H be a normal π-projective ss-supplemented Krasner hypermodule. Then H is amply ss-supplemented.

Proof. Let F and L be subhypermodules of H such that $H = F + L$. By the hypothesis, there exists a strong endomorphism α of H such that $\alpha(H) \leq F$ and $(1-\alpha)(H) \leq L$. Note that $(1-\alpha)(F) \leq F$. Let L' be an ss-supplement of F in H. Then $H = \alpha(H) + (1-\alpha)(H) = \alpha(H) + (1-\alpha)(F + L') \leq F + (1-\alpha)(L')$, so that $H = F + (1-\alpha)(L')$. Note that $(1-\alpha)(L') \leq L$. Let $x \in F \cap (1-\alpha)(L')$. Then, $x \in F$ and

$$x = (1-\alpha)(a) = a - \alpha(a)$$

for some $a \in L'$. It follows from $a = x + \alpha(a) \in F$ that $x \in (1-\alpha)(F \cap L')$. Since $F \cap L' \ll L'$

$$F \cap (1-\alpha)(L') = (1-\alpha)(F \cap L') \ll (1-\alpha)(L')$$

by Proposition 2.6 of (Talaee, 2013). Since $F \cap (1-\alpha)(L') = (1-\alpha)(F \cap L')$ is semisimple, $(1-\alpha)(L')$ is an ss-supplement of F in H. Thus, H is amply ss- supplement. ■

Corollary 4.19 Any subhypermodule of a normal projective ss-supplemented Krasner hypermodule is ss-supplemented.

Proposition 4.20 Let H be a normal projective hypermodule. Then H is ss-supplemented if and only if it is supplemented and $Rad(H) \subseteq Soc(H)$.

Proof. Since every normal projective hypermodule has small radical, the proof follows from Theorem 4.8. ■

5. CONCLUSION

The aim of this book chapter is to reveal the existence of the concept of ss-supplemented Krasner hypermodule over a Krasner hyperring R. In our study, firstly the concept of strong local Krasner hypermodule, which has an important place in the definition of ss-supplemented Krasner hypermodules, was introduced as a strong notion of supplemented Krasner hypermodule. The concepts of ss-supplemented Krasner hypermodules and amply ss-supplemented Krasner hypermodules were introduced. Every strongly local Krasner hypermodule is proved to be an ss-supplemented Krasner hypermodule. An example of a module that is amply supplenmented but not amply ss-supplemented Krasner hypermodule is given. It has been shown that π-projective ss-supplemented Krasner hypermodules are ss-supplemented Krasner hypermodules. Our results specialized some known results on (Hamzekolaee et al., 2021) and generalize of the notion of ss-supplemented modules in (Kaynar et al., 2020).

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Chapter 4

COMPACT EMBEDDING AND INCLUSION THEOREMS FOR WEIGHTED FUNCTION SPACES WITH WAVELET TRANSFORM

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1. INTRODUCTION

Wavelet theory is very popular topic and an alternative to time-frequency analysis. Many researchers study on wavelet theory (Daubechies, 1992; Gröchenig, 2001; Mallat, 1998). The parameters in wavelet theory are "*time*" x and "*scale*" s. "*Dilation operator*" D_s is given by $D_s f(t) = |s|^{-\frac{d}{2}} f\left(\frac{t}{s}\right)$ for all $t \in \mathbb{R}^d$, $0 \neq s \in \mathbb{R}$. It preserves the shape of $f: \mathbb{R}^d \to \mathbb{C}$, but it changes the scale. "*The continuous wavelet transform*" of a function f with respect to wavelet g is defined by

$$W_g f(x,s) = |s|^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(t)\, \overline{g\left(\frac{t-x}{s}\right)}\, dt$$

for $x \in \mathbb{R}^d$ and $0 \neq s \in \mathbb{R}$ (Gröchenig, 2001). The Wavelet transform is written as convolution $W_g f(x,s) = f * D_s g^*(x)$, where $g^*(t) = \overline{g(-t)}$. Also it is known that $W_g(T_z f) = T_{(z,0)} W_g f$ (Kulak & Gürkanlı, 2011). For $g_1, g_2 \in L^2(\mathbb{R}^d)$,

$$\int_0^\infty \frac{\left|\widehat{g_1(s\omega)}\widehat{g_2(s\omega)}\right|}{s} ds < \infty$$

and

$$\int_0^\infty \frac{\overline{\widehat{g_1(s\omega)}}\widehat{g_2(s\omega)}}{s} ds = K \text{ (independent of } \omega)$$

is called the "*wavelet admissibility condition*" (Daubechies, 1992; Gröchenig, 2001). If $g_1, g_2 \in L^2(\mathbb{R}^{\boldsymbol{d}})$ satisfy the admissibility condition, then

$$\int_0^\infty \int_{\mathbb{R}^d} W_{g_1} f_1(x,s) \overline{W_{g_2} f_2(x,s)} \frac{dxds}{s^{d+1}} = K\langle f_1, f_2 \rangle$$

for all $f_1, f_2 \in L^2(\mathbb{R}^d)$ (Daubechies, 1992; Gröchenig, 2001). If

$g_1, g_2 \in L^2(\mathbb{R}^d)$ satisfy admissibility condition, then $f \in L^2(\mathbb{R}^d)$ is reconstructed from it's the wavelet transform by

$$f = \frac{1}{K} \int_{\mathbb{R}^d} \int_0^\infty W_{g_1} f(x,s) T_x D_s\, g_2 \frac{dxds}{s^{d+1}}$$

(Daubechies, 1992; Gröchenig, 2001). In this paper the "*weight function* ω" is positive real valued, measurable and locally bounded on $\mathbb{R}^d$ which satisfies

$$\omega(x) \geq 1, \omega(x+y) \leq \omega(x)\omega(y)$$

for all $x, y \in \mathbb{R}^d$ (Reiter, 1968). A weight $\omega(x) = (1+|x|)^a$ is called "*weight of polynomial type*" such that $x \in \mathbb{R}^d$ and $a \geq 0$. If the weights ω_1 and ω_2 satisfy the condition $\omega_1(x) \leq C\omega_2(x)$, $(C > 0)$ for all $x \in \mathbb{R}^d$, we denote with this symbol $\omega_1 \prec \omega_2$. Also if the weight functions ω_1 and ω_2 are equivalent, we write that $\omega_1 \approx \omega_2$ if and only if $\omega_1 \prec \omega_2$ and $\omega_2 \prec \omega_1$.

The function

$$\lambda_f(y) = \omega(\{x \in \mathbb{R}^d : |f(x)| > y\}) = \int_{\{x \in \mathbb{R}^d : |f(x)| > y\}} \omega(x) dx$$

is said that "*distribution function*", (Blozinski, 1972; Hunt, 1966; Hunt & Kurtz, 1983). The "*rearangement function*" is given by

$$f^*(t) = inf\{y > 0: \lambda_f(y) \leq t\} = sup\{y > 0: \lambda_f(y) > t\}$$

for $t \geq 0$ (Blozinski, 1972; Hunt, 1966; Hunt & Kurtz, 1983). Also, the "*average function*" is defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$$

for $t > 0$ (Blozinski, 1972; Hunt, 1966; Hunt & Kurtz, 1983). The "*weighted Lorentz space* $L(p,q,\omega d\mu)(\mathbb{R}^d)$" is a vector space of measurable functions f on $\mathbb{R}^{\boldsymbol{d}}$ such that $\|f\|^*_{pq,\omega} < \infty$, where

$$\|f\|^*_{pq,\omega} = \left(\frac{q}{p}\int_0^\infty t^{\frac{q}{p}-1}\left(f^*(t)\right)^q dt\right)^{\frac{1}{q}}, \text{ if } 0 < p,q < \infty$$

$$\|f\|^*_{pq,\omega} = \sup_{t>0} t^{\frac{1}{p}} f^*(t), \text{ if } 0 < p < q = \infty.$$

This space is a normed space with the following norm by (Blozinski, 1972; Hunt, 1966; Hunt & Kurtz, 1983; Duyar & Gürkanlı, 2003)

$$\|f\|_{pq,\omega} = \left(\frac{q}{p}\int_0^\infty t^{\frac{q}{p}-1}\left(f^{**}(t)\right)^q dt\right)^{\frac{1}{q}}, \text{ if } 0 < p,q < \infty$$

$$\|f\|_{pq,\omega} = \sup_{t>0} t^{\frac{1}{p}} f^*(t), \text{ if } 0 < p < q = \infty.$$

2. THE SPACE $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$

In this chapter, we assume that the scales of wavelet transform is fixed. The space $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$ is the vector space of functions $f \in L^p_{\omega_1}(\mathbb{R}^d)$ such that their wavelet transforms $W_g f$ in $L(q,r,\omega_2 d\mu)(\mathbb{R}^d)$.

Definition 2.1 Let $1 \leq p, q, r < \infty$ and ω_1, ω_2 be weight functions on $\mathbb{R}^d$. Assume that $0 \neq g \in S(\mathbb{R}^d)$ which denotes space of complex-valued continuous functions on $\mathbb{R}^d$ rapidly decreasing at infinity. For $s \in \mathbb{R}^+$, we set

$$L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d) = \{f \in L^p_{\omega_1}(\mathbb{R}^d) : W_g f \in L(q,r,\omega_2 d\mu)(\mathbb{R}^d)\}.$$

The space $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$ is a normed Banach space with this norm

$$\|f\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}} = \|f\|_{p,\omega_1} + \|W_g f\|_{qr,\omega_2}.$$

Theorem 2.2 a) The space $C_c^\infty(\mathbb{R}^d)$ which denotes space of infinitely differentiable complex-valued functions with compact supported on $\mathbb{R}^d$, is dense in $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$.

b) Assume that ω_2 is weight function of polynomial type. Then $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$ is dense in $L^p_{\omega_1}(\mathbb{R}^d)$.

Proof. a) For arbitrary $h \in C_c^\infty(\mathbb{R}^d)$, we have $h \in L^p_{\omega_1}(\mathbb{R}^d)$. It's known that $L(q,r,\omega_2 d\mu)(\mathbb{R}^d)$ is a Banach module over $L^1_{\omega_2}(\mathbb{R}^d)$ (Duyar & Gürkanlı, 2003). Then

$$\|W_g h\|_{qr,\omega_2} = \|h * D_s g^*\|_{qr,\omega_2} \leq \|h\|_{qr,\omega_2} \|D_s g^*\|_{1,\omega_2} < \infty$$

and $W_g h \in L(q,r,\omega_2 d\mu)(\mathbb{R}^d)$ is written. So we have $h \in L(q,r,\omega_2 d\mu)(\mathbb{R}^d)$. That means $C_c^\infty(\mathbb{R}^d) \subset L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$. Now let be $h \in L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$. So $h \in L^p_{\omega_1}(\mathbb{R}^d)$ and $W_g h \in L(q,r,\omega_2 d\mu)(\mathbb{R}^d)$. Also since $C_c^\infty(\mathbb{R}^d)$ is dense in these spaces, there exist $(h_n)_{n\in\mathbb{N}}$, $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^d)$ such that

$$\|h_n - h\|_{p,\omega_1} \to 0, \|f_n - W_g h\|_{qr,\omega_2} \to 0.$$

By using the subsequence property, we find a subsequence

$$(f_{n_k})_{n_k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^d)$$

such that $f_{n_k} = W_g h_{n_k}$ and $\|f_{n_k} - W_g h\|_{qr,\omega_2} \to 0$, where

$$\left(h_{n_k}\right)_{n_k\in\mathbb{N}} \subset (h_n)_{n\in\mathbb{N}}.$$

Therefore $\left\|h_{n_k} - f\right\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}} \to 0$ and $\left(h_{n_k}\right)_{n_k\in\mathbb{N}} \subset C_c^\infty\left(\mathbb{R}^d\right)$. Hence

$$\overline{C_c^\infty(\mathbb{R}^d)} = L_s(W)^{p,q,r}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right).$$

b) Since ω_2 is weight of polynomial type, we have that $D_s g^* \in L^1_{\omega_2}\left(\mathbb{R}^d\right)$. Let be $f \in C_c\left(\mathbb{R}^d\right) \subset L^p_{\omega_1}\left(\mathbb{R}^d\right)$. Then we write

$$\left\|W_g f\right\|_{qr,\omega_2} = \|f * D_s g^*\|_{qr,\omega_2} \leq \|f\|_{qr,\omega_2}\|D_s g^*\|_{1,\omega_2} < \infty.$$

Hence $C_c\left(\mathbb{R}^d\right) \subset L_s(W)^{p,q,r}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right) \subset L^p_{\omega_1}\left(\mathbb{R}^d\right)$. Since $\overline{C_c(\mathbb{R}^d)} = L^p_{\omega_1}\left(\mathbb{R}^d\right)$, we find $\overline{L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)} = L^p_{\omega_1}\left(\mathbb{R}^d\right)$.

Theorem 2.3 a) $L_s(W)^{p,q,r}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$ is invariant under translations.

b) The mapping $f \to T_z f$ is continuous from $L_s(W)^{p,q,r}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$ into $L_s(W)^{p,q,r}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$ for every $f \in L_s(W)^{p,q,r}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$ and fixed $z \in \mathbb{R}^d$.

c) The mapping $z \to T_z f$ is continuous from $\mathbb{R}^d$ into $L_s(W)^{p,q,r}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$.

Proof. a) Take any $f \in L_s(W)^{p,q,r}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$. So

$$f \in L^p_{\omega_1}\left(\mathbb{R}^d\right) \text{ and } W_g f \in L(q,r,\omega_2 d\mu)\left(\mathbb{R}^d\right).$$

Since $\|T_z f\|_{p,\omega_1} \leq \omega_1(z)\|f\|_{p,\omega_1}$, we write $T_z f \in L^p_{\omega_1}\left(\mathbb{R}^d\right)$ for all $z \in \mathbb{R}^d$ (Fischer et al., 1996). Also by using the equality $W_g(T_z f) = T_{(z,0)} W_g f$, we find that

$$\left\|W_g(T_z f)\right\|_{qr,\omega_2} \leq \omega_2(z)^{\frac{1}{q}}\left\|W_g f\right\|_{qr,\omega_2} \tag{4.1}$$

for all $z \in \mathbb{R}^d$ (Duyar & Gürkanlı, 2003). Then we have

$$\|T_z f\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}} \leq \omega_1(z)\|f\|_{p,\omega_1} + \omega_2(z)^{\frac{1}{q}}\left\|W_g f\right\|_{qr,\omega_2}$$

and so $T_z f \in L_s(W)^{p,q,r}_{\omega_1,\omega_2}\left(\mathbb{R}^d\right)$.

b) Take arbitrary $f \in L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$ and $\varepsilon > 0$. Let $\delta > 0$ such that

$$\delta = \frac{\varepsilon}{\omega_1(z) + \omega_2(z)^{\frac{1}{q}}}.$$

If $\|f\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}} < \delta$, then $\|f\|_{p,\omega_1} \leq \|f\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}} < \delta$ and

$$\|f\|_{qr,\omega_2} \leq \|f\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}} < \delta.$$

Also since $\left\|W_g(T_z f)\right\|_{qr,\omega_2} \leq \omega_2(z)^{\frac{1}{q}}\left\|W_g f\right\|_{qr,\omega_2}$, we find that

$$\|T_z f\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}} \leq \omega_1(z)\|f\|_{p,\omega_1} + \omega_2(z)^{\frac{1}{q}}\left\|W_g f\right\|_{qr,\omega_2}$$

$$\leq \delta\left(\omega_1(z) + \omega_2(z)^{\frac{1}{q}}\right) < \varepsilon.$$

c) Given $f \in L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$. We know that the translation mapping is continuous from $\mathbb{R}^d$ into $L^p_{\omega_1}(\mathbb{R}^d)$ (Fischer et al., 1996). For any $\varepsilon > 0$, if $\|u - v\| < \delta_1$ for $u, v \in \mathbb{R}^d$, then there exists $\delta_1 > 0$ such that

$$\|T_u f - T_v f\|_{p,\omega_1} < \frac{\varepsilon}{2}.$$

Also since the translation mapping is continuous from $\mathbb{R}^d$ into $L(q, r, \omega_2 d\mu)(\mathbb{R}^d)$ (Duyar & Gürkanlı, 2003), for same $\varepsilon > 0$, there exists $\delta_2 > 0$ such that if $\|u - v\| < \delta_2$ for all $u, v \in \mathbb{R}^d$, then

$$\left\|W_g(T_u f - T_v f)\right\|_{qr,\omega_2} = \left\|W_g(T_u f) - W_g(T_v f)\right\|_{qr,\omega_2}$$

$$= \left\|T_{(u,0)} W_g f - T_{(v,0)} W_g f\right\|_{qr,\omega_2} < \frac{\varepsilon}{2}.$$

Let $\delta = min\{\delta_1, \delta_2\}$. If $\|u - v\| < \delta_2$, then

$$\|T_u f - T_v f\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}}$$

$$= \|T_u f - T_v f\|_{p,\omega_1} + \left\|W_g(T_u f - T_v f)\right\|_{qr,\omega_2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \blacksquare$$

Theorem 2.4 $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$ is a Banach function space.

Proof. Let be $f \in L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$. There exists $C > 0$ such that

$$\int_{\mathbb{R}^d} |f(x)| dx \leq C\|f\|_p$$

where a compact subset $K \subset \mathbb{R}^d$. Then

$$\int_{\mathbb{R}^d} |f(x)| dx \leq C\left\{\|f\|_{p,\omega_1} + \left\|W_g f\right\|_{qr,\omega_2}\right\} = C\|f\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}}.$$

Also if we use that $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$ is a Banach space and the last inequality, we find that this space is a Banach function space. ■

Theorem 2.5 Let $\omega_2 \leq \omega_1$. The space $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$ is an essential Banach module over $L^1_{\omega_1}(\mathbb{R}^d)$.

Proof. Let be $f \in L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$ and $h \in L^1_{\omega_1}(\mathbb{R}^d)$. The we have

$$\|f * h\|_{p,\omega_1} \leq \|f\|_{p,\omega_1}\|h\|_{1,\omega_1} \tag{4.2}$$

and

$$.\left\|W_g(f * h)\right\|_{qr,\omega_2} = \|(f * h) * D_s g^*\|_{qr,\omega_2}$$

$$\leq \|h\|_{1,\omega_2}\|f * D_s g^*\|_{qr,\omega_2} \leq \|h\|_{1,\omega_1}\left\|W_g f\right\|_{qr,\omega_2}. \tag{4.3}$$

So by (4.2) and (4.3), we have

$$\|f * h\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}} = \|f * h\|_{p,\omega_1} + \left\|W_g(f * h)\right\|_{qr,\omega_2}$$

$$\leq \|h\|_{1,\omega_1}\left\{\|f\|_{p,\omega_1} + \left\|W_g f\right\|_{qr,\omega_2} = \|h\|_{1,\omega_1}\|f\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}}\right\}.$$

Therefore we find that $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$ is a Banach module over $L^1_{\omega_1}(\mathbb{R}^d)$ (Larsen, 1973; Liu & Rooij, 1969). Now we show that

$$L^1_{\omega_1}(\mathbb{R}^d) * L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d) = L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d).$$

It is known that $L^1_{\omega_1}(\mathbb{R}^d)$ has a bounded approximate identity (Gaudry, 1969). Let U be neighbourhood of the unit element of $\mathbb{R}^d$. We can choose an approximate identity $(e_\alpha)_{\alpha\in I}$ which is positive bounded and $suppe_\alpha \subset U$, $\|e_\alpha\|_1 = 1$ for all $\alpha \in I$. Take $h \in L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$. Then

$$\left\|e_{\alpha_0} * h - h\right\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}}$$

$$= \left\|\int_{\mathbb{R}^d} e_{\alpha_0}(z)\, T_z h(y)dz - \int_{\mathbb{R}^d} e_{\alpha_0}(z)\, h(y)dz\right\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}}$$

$$= \left\|\int_{\mathbb{R}^d} e_{\alpha_0}(z)\left(T_z h(y) - h(y)\right)dz\right\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}}$$

$$\leq \int_{\mathbb{R}^d} e_{\alpha_0}(z)\, \|T_z h - h\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}} dz.$$

where fixed $\alpha_0 \in I$. Since the translation mapping $z \to T_z f$ is continious from $\mathbb{R}^d$ into $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$ by Theorem 2.3, given any $\varepsilon > 0$, we say that

$$\|T_z h - h\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}} < \varepsilon.$$

Therefore we get

$$\left\|e_{\alpha_0} * h - h\right\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}} \leq \int_{\mathbb{R}^d} e_{\alpha_0}(z)\, \varepsilon dz = \varepsilon.$$

That means $L^1_{\omega_1}(\mathbb{R}^d) * L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d) = L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$. Using Module Factorization Theorem (Wang, 1977), we obtain that $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$ is an essential Banach module over $L^1_{\omega_1}(\mathbb{R}^d)$.

Corollary 2.6 Let $\omega_2 \leq \omega_1$. Assume that $(e_\alpha)_{\alpha\in I}$ is an approximate identity in $L^1_{\omega_1}(\mathbb{R}^d)$. Then $(e_\alpha)_{\alpha\in I}$ is an approximate identity of the space $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$.

Proof. From the Theorem 2.5 and (Doran & Wichmann, 1979), the proof is easily achieved. ■

3. INCLUSION PROPERTIES OF THE SPACE $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$

Theorem 3.1 If $L_s(W)^{p,q,r}_{\omega_1,\omega_3}(\mathbb{R}^d) \subset L_s(W)^{p,q,r}_{\omega_2,\omega_4}(\mathbb{R}^d)$, then $L_s(W)^{p,q,r}_{\omega_1,\omega_3}(\mathbb{R}^d)$ is a Banach space under the norm

$$\|f\|_L = \|f\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_3}} + \|f\|_{L_s(W)^{p,q,r}_{\omega_2,\omega_4}}.$$

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\left(L_s(W)^{p,q,r}_{\omega_1,\omega_3}(\mathbb{R}^d), \|.\|_L\right)$. So $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the spaces $\left(L_s(W)^{p,q,r}_{\omega_1,\omega_3}(\mathbb{R}^d), \|.\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_3}}\right)$ and $\left(L_s(W)^{p,q,r}_{\omega_2,\omega_4}(\mathbb{R}^d), \|.\|_{L_s(W)^{p,q,r}_{\omega_2,\omega_4}}\right)$. Since these spaces are Banach spaces, there exist $f \in L_s(W)^{p,q,r}_{\omega_2,\omega_4}(\mathbb{R}^d)$ and $h \in L_s(W)^{p,q,r}_{\omega_1,\omega_3}(\mathbb{R}^d)$ such that

$$\|f_n - f\|_{L_s(W)^{p,q,r}_{\omega_2,\omega_4}} \to 0, \|f_n - h\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_3}} \to 0$$

From the inequalities $\|.\|_p \le \|.\|_{L_s(W)^{p,q,r}_{\omega_2,\omega_4}}$ and $\|.\|_p \le \|.\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_3}}$, we have $\|f_n - f\|_p \to 0$, $\|f_n - h\|_p \to 0$. So since

$$\|f - h\|_p \le \|f_n - f\|_p + \|f_n - h\|_p,$$

we find that $\|f - h\|_p = 0$ and $f = h$. Hence

$$\|f_n - f\|_L \to 0 \text{ and } f \in \left(L_s(W)^{p,q,r}_{\omega_1,\omega_3}(\mathbb{R}^d), \|.\|_L\right). \qquad \blacksquare$$

Theorem 3.2 If $\omega = max\{\omega_1, \omega_3\}$ and $m = max\{\omega_2, \omega_4\}$, then we have

$$L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d) \cap L_s(W)^{p,q,r}_{\omega_3,\omega_4}(\mathbb{R}^d) = L_s(W)^{p,q,r}_{\omega,m}(\mathbb{R}^d)$$

Proof. Let $f \in L_s(W)^{p,q,r}_{\omega,m}(\mathbb{R}^d)$ be arbitrary. Then

$$\|f\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}} = \|f\|_{p,\omega_1} + \left\|W_g f\right\|_{qr,\omega_2} \le \|f\|_{p,\omega} + \left\|W_g f\right\|_{qr,m} < \infty$$

and so $f \in L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$. Similarly $f \in L_s(W)^{p,q,r}_{\omega_3,\omega_4}(\mathbb{R}^d)$ is written. Thus we have

$$L_s(W)^{p,q,r}_{\omega,m}(\mathbb{R}^d) \subset L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d) \cap L_s(W)^{p,q,r}_{\omega_3,\omega_4}(\mathbb{R}^d). \qquad (4.4)$$

Conversely let $f \in L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d) \cap L_s(W)^{p,q,r}_{\omega_3,\omega_4}(\mathbb{R}^d)$. Since the assumption $\omega = max\{\omega_1, \omega_3\}$ and $m = max\{\omega_2, \omega_4\}$, we achieve

$f \in L_s(W)^{p,q,r}_{\omega,m}(\mathbb{R}^d)$.

So we have

$$L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d) \cap L_s(W)^{p,q,r}_{\omega_3,\omega_4}(\mathbb{R}^d) \subset L_s(W)^{p,q,r}_{\omega,m}(\mathbb{R}^d). \tag{4.5}$$

Hence by (4.4) and (4.5), we get

$$L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d) \cap L_s(W)^{p,q,r}_{\omega_3,\omega_4}(\mathbb{R}^d) = L_s(W)^{p,q,r}_{\omega,m}(\mathbb{R}^d).$$ ■

Theorem 3.3 If $\omega_3 \prec \omega_1$ and $\omega_4 \prec \omega_2$, then

$$L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d) \subset L_s(W)^{p,q,r}_{\omega_3,\omega_4}(\mathbb{R}^d)$$

for all $f \in L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$.

Proof. Since the assumptions, there exist $C_1, C_2 > 0$ such that

$$\omega_1(t) \leq C_1 \omega_3(t)$$

and $\omega_4(t) \leq C_2 \omega_2(t)$ for all $t, z \in \mathbb{R}^d$. Let $f \in L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$. So

$f \in L^p_{\omega_1}(\mathbb{R}^d)$ and $W_g f \in L(q, r, \omega_2 d\mu)(\mathbb{R}^d)$.

Then

$$\|f\|_{p,\omega_3} \leq C_1 \|f\|_{p,\omega_1} \text{ and } \left\|W_g f\right\|_{p,\omega_4} \leq C_3 \left\|W_g f\right\|_{qr,\omega_2}.$$

We find $f \in L_s(W)^{p,q,r}_{\omega_3,\omega_4}(\mathbb{R}^d)$. Hence we obtain

$$L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d) \subset L_s(W)^{p,q,r}_{\omega_3,\omega_4}(\mathbb{R}^d).$$ ■

Theorem 3.4 If Suppose that $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d) \subset L_s(W)^{p,q,r}_{\omega_3,\omega_4}(\mathbb{R}^d)$. Then there exists a $C > 0$ such that

$$\|f\|_{L_s(W)^{p,q,r}_{\omega_3,\omega_4}} \leq C\|f\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}}$$

for each $f \in L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$.

Proof. If we endow the space $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$ with the norm

$$\|f\|_L = \|f\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}} + \|f\|_{L_s(W)^{p,q,r}_{3,\omega_4}}$$

then the space $\left(L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d), \|.\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}}\right)$ is a Banach space by from Theorem 3.1. Also using Closed Graph Theorem, there exists a $C > 0$ such that

$$\|f\|_{L_s(W)^{p,q,r}_{\omega_3,\omega_4}} \leq C\|f\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}}$$

for each $f \in L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$. ■

Theorem 3.5 For every $0 \neq f \in L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$, there exists $C(f) > 0$ such that

$$C(f)\omega_1(z) \leq \|T_z f\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}} \leq \left(\omega_1(z) + \omega_2(z)^{\frac{1}{q}}\right)\|f\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}}$$

Proof. Take any $0 \neq f \in L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$. There exists $C(f) > 0$ such that

$$C(f)\omega_1(z) \leq \|T_z f\|_{p,\omega_1} \leq \omega_1(z)\|f\|_{p,\omega_1} \tag{4.6}$$

(Feichtinger & Gürkanlı, 1990).

On the other hand, using the inequalities (4.1) and (4.6), we obtain

$$C(f)\omega_1(z) \leq \|T_z f\|_{p,\omega_1} + \left\|W_g T_z f\right\|_{qr,\omega_2}$$

$$\leq \omega_1(z)\|f\|_{p,\omega_1} + \omega_2(z)^{\frac{1}{q}}\left\|W_g f\right\|_{qr,\omega_2}$$

$$\leq \omega_1(z)\|f\|_{p,\omega_1} + \omega_2(z)^{\frac{1}{q}}\left\|W_g f\right\|_{qr,\omega_2}$$

$$\leq \omega_1(z)\|f\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}} + \omega_2(z)^{\frac{1}{q}}\|f\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}}$$

$$= \left(\omega_1(z) + \omega_2(z)^{\frac{1}{q}}\right) \|f\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}}.$$

This completes proof. ■

4. COMPACT EMBEDDINGS OF THE SPACE $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$

Lemma 4.1 Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$. If $(f_n)_{n\in\mathbb{N}}$ converges to zero in $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$, then

$$\int_{\mathbb{R}^d} f_n(x)k(x)dx \to 0$$

for $n \to \infty$ and for all $k \in C_c(\mathbb{R}^d)$.

Proof. Assume that $k \in C_c(\mathbb{R}^d)$ and $\frac{1}{p} + \frac{1}{r} = 1$. So

$$\left|\int_{\mathbb{R}^d} f_n(x)k(x)dx\right| \leq \|k\|_r \|f_n\|_p \leq \|k\|_r \|f_n\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}} \tag{4.7}$$

is obtained. From assumption and (4.7), we find

$$\int_{\mathbb{R}^d} f_n(x)k(x)dx \to 0$$

for $n \to \infty$ and for all $k \in C_c(\mathbb{R}^d)$.

In the following theorem, we use the similar technique of proof in (Gürkanlı, 2008).

Theorem 4.2 Assume that $\omega_1,\ \omega_2$ are weight functions of polynomial type and u is weight function on $\mathbb{R}^d$. If $u \prec \omega_1$ and $\frac{u(x)}{\omega_1(x)+\ \omega_2(x,s)} \nrightarrow 0$ for $x \to \infty$, then the embedding of the space $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$ into $L^p_u(\mathbb{R}^d)$ is never compact.

Proof. From the assumption $u \prec \omega_1$, we say that there exists $C_1 > 0$ such that $u(x) \leq C_1\omega_1(x)$ for all $x \in \mathbb{R}^d$. This implies

$$L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d) \subset L^p_u(\mathbb{R}^d).$$

Suppose that $(t_n)_{n\in\mathbb{N}}$ is a sequence with $t_n \to \infty$ as $n \to \infty$ in $\mathbb{R}^d$. Also since the assumption $\frac{u(x)}{\omega_1(x)+\omega_2(x,s)}$ does not tend to zero as $x \to \infty$, there exists $\delta > 0$ such that $\frac{u(x)}{\omega_1(x)+\omega_2(x,s)} \geq \delta > 0$ for $x \to \infty$. Fixed $f \in L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$ and $t_0 \in \mathbb{R}_+$, we define a sequence $(f_n)_{n\in\mathbb{N}}$ such that

$$f_n = \left(\omega_1(t_n) + \omega_2(t_n, t_0)\right)^{-1} T_{t_n} f.$$

Also Lemma 4.1, we have

$$\|f_n\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}} = \left\|\left(\omega_1(t_n) + \omega_2(t_n, t_0)\right)^{-1} T_{t_n} f\right\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}}$$

$$= \left(\omega_1(t_n) + \omega_2(t_n, t_0)\right)^{-1} \left\|T_{t_n} f\right\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}}$$

$$\leq \left(\omega_1(t_n) + \omega_2(t_n, t_0)\right)^{-1} \left(\omega_1(t_n) + \omega_2(t_n, t_0)\right) \|f\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}}$$

$$= \|f\|_{L_s(W)^{p,q,r}_{\omega_1,\omega_2}}.$$

That means this sequence is bounded in $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$. On the other hand, we will show that there wouldn't exist norm convergence subsequence of $(f_n)_{n\in\mathbb{N}}$ in $L^p_u(\mathbb{R}^d)$. Then

$$\left|\int_{\mathbb{R}^d} f_n(x) k(x) dx\right|$$

$$\leq \frac{1}{\omega_1(t_n) + \omega_2(t_n, t_0)} \int_{\mathbb{R}^d} \left|T_{t_n} f(x)\right| |k(x)| dx$$

$$\leq \frac{1}{\omega_1(t_n) + \omega_2(t_n, t_0)} \|k\|_s \left\|T_{t_n} f\right\|_p$$

$$= \frac{1}{\omega_1(t_n) + \omega_2(t_n, t_0)} \|k\|_s \left\|T_{t_n} f\right\|_p \tag{4.8}$$

is written where $\frac{1}{p}+\frac{1}{r}=1$ for all $k\in C_c(\mathbb{R}^d)$. The last inequality (4.8) tends zero for $n\to\infty$, then we find

$$\int_{\mathbb{R}^d} f_n(x)k(x)dx \to 0.$$

By Lemma 4.1, the only possible limit of $(f_n)_{n\in\mathbb{N}}$ in $L_u^p(\mathbb{R}^d)$ is zero. Using

$$\left\|T_{t_n}f\right\|_p \approx u(t_n).$$

So there exists $C_1, C_2 > 0$ such that

$$C_1u(t_n) \le \left\|T_{t_n}f\right\|_{p,u} \le C_2u(t_n). \tag{4.9}$$

Using the inequality (4.9), we have

$$\|f_n\|_{p,u} = \left\|\left(\omega_1(t_n)+\ \omega_2(t_n,t_0)\right)^{-1}T_{t_n}f\right\|_{p,u}$$

$$= \left(\omega_1(t_n)+\ \omega_2(t_n,t_0)\right)^{-1}\left\|T_{t_n}f\right\|_{p,u}$$

$$\ge C_1u(t_n)\left(\omega_1(t_n)+\ \omega_2(t_n,t_0)\right)^{-1}. \tag{4.10}$$

Also since

$$\frac{u(t_n)}{\omega_1(t_n)+\ \omega_2(t_n,s)} \ge \delta > 0$$

for all t_n, and from the inequality (4.10), we obtain

$$\|f_n\|_{p,u} \ge C_1u(t_n)\left(\omega_1(t_n)+\ \omega_2(t_n,t_0)\right)^{-1} > \delta C_1 > 0.$$

Therefore we say that there would not be possible to find norm convergent subsequence of $(f_n)_{n\in\mathbb{N}}$ in $L_u^p(\mathbb{R}^d)$. The proof is completed. ■

Theorem 4.3 Assume that $\omega_1,\ \omega_2$ are weight functions of polynomial type and $\omega_3,\ \omega_4$ are any weight functions. If $\omega_3 \prec \omega_1,\ \omega_4 \prec\ \omega_2$ and

$$\frac{\omega_3(x)}{\omega_1(x)+\ \omega_2(x,s)} \nrightarrow 0$$

for $x \to \infty$, then the embedding of the space $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$ into $L_s(W)^{p,q,r}_{\omega_3,\omega_4}(\mathbb{R}^d)$ is never compact.

Proof. From the assumptions $\omega_3 \prec \omega_1$ and $\omega_4 \prec \omega_2$ and by from Theorem 3.3, we have that $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d) \subset L_s(W)^{p,q,r}_{\omega_3,\omega_4}(\mathbb{R}^d)$. The unit map is a continuous from $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$ into $L_s(W)^{p,q,r}_{\omega_3,\omega_4}(\mathbb{R}^d)$. Suppose that the unit map is compact. Let $(f_n)_{n\in\mathbb{N}}$ in $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$ be an arbitrary bounded sequence. If there exists convergent subsequence of $(f_n)_{n\in\mathbb{N}}$ in $L_s(W)^{p,q,r}_{\omega_3,\omega_4}(\mathbb{R}^d)$, this sequence also converges in $L^p_{\omega_3}(\mathbb{R}^d)$. But this is not possible by Theorem 4.2. This completes the proof.

5. CONCLUSION

The wavelet transform acting like a microscope gives us local information of signals at any time and any size. Thanks to this property of the wavelet transform, wavelet theory is an important field for harmonic analysis, applied mathematics, signal analysis. In this chapter, using the wavelet transform, the space $L_s(W)^{p,q,r}_{\omega_1,\omega_2}(\mathbb{R}^d)$ is defined and some fundamental properties are considered. Then the inclusion and compact embeddings theorems are proved.

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Chapter 5

BILINEAR MULTIPLIERS OF FUNCTION SPACES WITH WIGNER TRANSFORM

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1. INTRODUCTION

Throughout this work $S(\mathbb{R})$ denotes the space of complex-valued continuous functions on $\mathbb{R}$ rapidly decreasing at infinity, respectively. Assume that f is a complex-valued measurable function on $\mathbb{R}$. The space $\mathrm{L}^{\mathrm{p}}(\mathbb{R})$, $(1\leq p \leq \infty)$ denotes the usual "*Lebesgue space*" (Reiter, 1968). A continuous and measurable function ω satisfying $1 \leq \omega(x)$ and $\omega(x+y) \leq \omega(x)\omega(y)$ for $x, y \in \mathbb{R}$ will be called a "*weight function*" on $\mathbb{R}$. Let $a \geq 0$. The function

$$\omega(x,y) = (1 + |x| + |y|)^a$$

which is defined on $\mathbb{R}^2$ is called "w*eight of polynomial*" *type* (Gasquet & Witomski, 1999). For $1\leq p \leq \infty$, *the weighted* "*Lebesgue space*" is defined by $L^p_\omega(\mathbb{R}) = \{f : f\omega \in L^p(\mathbb{R})\}$ (Reiter, 1968). The translation, character and dilation operators T_a, M_a and D_s are given by

$$T_a f(x) = f(x-a),\ M_a f(x) = e^{2\pi i x a} f(x),\ D_s f(x) = |s|^{-\frac{1}{2}} f\left(\frac{x}{s}\right)$$

respectively for $a, x \in \mathbb{R}$, $s \neq 0$. For $f \in L^1(\mathbb{R})$, the Fourier transform is denoted by $\hat{f}$. Also the Fourier transform of the dilation operator is

$$\widehat{D_s f}(\xi) = D_{s^{-1}} \hat{f}(\xi)$$

for all $\xi \in \mathbb{R}$ (Gröchenig, 2001). The set $M(\mathbb{R})$ denotes the space of bounded regular Borel measures. Also we denote by $M(\omega)$ the space of $\mu \in M(\mathbb{R})$ such that $\|\mu\|_{\omega} = \int_{\mathbb{R}} \omega d|\mu| < \infty$. For $\mu \in M(\omega)$, the "*Fourier-Stieljes transform*" is denoted by $\hat{\mu}$ (Rudin, 1962).

Let $0 \neq g \in L^2(\mathbb{R})$ be window function. The "*Gabor transform*" (short-time Fourier transform) of a function $f \in L^2(\mathbb{R})$ with respect to g is given by

$$V_g f(x,w) = \int_{\mathbb{R}} f(y)\overline{g(y-x)}e^{-2\pi i y w}dy$$

for $x, w \in \mathbb{R}$ (Gröchenig, 2001). For $\tau \in (0,1)$, the "$\tau -$*short-time Fourier transform*" of f with respect to g is given by $V_g^{\tau} f(x,w) = V_g f\left(\frac{x}{1-\tau}, \frac{w}{\tau}\right)$ for $x, w \in \mathbb{R}$ (Boggiatto et al., 2007:235-249).

The *"cross-Wigner distribution"* of $f, g \in L^2(\mathbb{R})$ is defined by

$$W(f,g)(x,w) = \int_{\mathbb{R}} f\left(x+\frac{y}{2}\right)\overline{g\left(x-\frac{y}{2}\right)}e^{-2\pi i y w}dy$$

for $x, w \in \mathbb{R}$. The cross-Wigner distribution which is a quadratic time-frequency representation, gives us information about the amount of signal energy during the any time period the energy density in time-frequency plane. For $\tau \in (0,1)$, the $\tau -$*Wigner transform* is defined by

$$W_{\tau}(f,g)(x,w) = \int_{\mathbb{R}} f(x+\tau y)\overline{g(x-(1-\tau)y)}e^{-2\pi i y w}dy$$

for $x, w \in \mathbb{R}$ (Gröchenig, 2001). Also the $\tau -$Wigner transform has the following relation with the $\tau -$short-time Fourier transform (Kulak & Ömerbeyoğlu, 2021:188-200).

$$W_{\tau}(f,g)(x,w) = e^{\frac{2\pi i x w}{\tau}} \frac{1}{\sqrt{\tau(1-\tau)}} V^{\tau}_{D_{\frac{\tau}{\tau-1}}g} f(x,w)$$

Let ω, ϑ be weight functions and let be $1 \le p, r < \infty$, $\tau \in (0,1)$. The space $CW_{\omega,\vartheta}^{p,r,\tau}(\mathbb{R})$ consists of all $f \in L_{\omega}^{p}(\mathbb{R})$ such that their $\tau -$Wigner transforms $W_{\tau}(f,.)$ are in $L_{\vartheta}^{r}(\mathbb{R}^2)$. This space is equipped with sum norm

$$\|f\|_{CW_{\omega,\vartheta}^{p,r,\tau}} = \|f\|_{p,\omega} + \|W_{\tau}(f,.)\|_{r,\vartheta}.$$

The space $CW_{\omega,\vartheta}^{p,r,\tau}(\mathbb{R})$ is a Banach space with this norm (Kulak & Ömerbeyoğlu, 2021:188-200).

2. THE BILINEAR MULTIPLIERS THEORY FOR FUNCTION SPACES WITH WIGNER TRANSFORM

Let $1 \le p_1, r_1, p_2, r_2 p_3, r_3 < \infty$, $\tau_1, \tau_{2,} \tau_3 \in (0,1)$ and $\omega_1, \vartheta_1, \omega_2, \vartheta_2\ \omega_3, \vartheta_3$ be weight functions. Suppose that $\omega_1, \vartheta_1, \omega_2, \vartheta_2$ are weight functions of polynomial type and $m(\xi, \eta)$ is a bounded, measurable function on $\mathbb{R}^2$. Define

$$B_m(f,g)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi)\hat{g}\,(\eta) m(\xi,\eta) e^{2\pi i \langle \xi+\eta, x\rangle} d\xi d\eta$$

for all $f, g \in S(\mathbb{R})$. m EK3 said to be a bilinear multiplier on $\mathbb{R}$ of type

$$CW\big(p_1, r_{1,}\omega_1, \vartheta_{1,}\tau_1; p_2, r_2, \omega_2, \vartheta_2\ , \tau_{2,}; p_3, r_3, \omega_3, \vartheta_3, \tau_3\big)$$

(shortly $CW\big(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i\big)$), if there exists C>0 such that

$$\|B_m(f,g)\|_{CW_{\omega_3,\vartheta_3}^{p_3,r_3,\tau_3}} \le C\|f\|_{CW_{\omega_1,\vartheta_1}^{p_1,r_1,\tau_1}} \|g\|_{CW_{\omega_2,\vartheta_2}^{p_2,r_2,\tau_2}}$$

for all $f, g \in S(\mathbb{R})$. That means B_m extends to a bounded bilinear operator from

$CW_{\omega_1,\vartheta_1}^{p_1,r_1,\tau_1}(\mathbb{R}) \times CW_{\omega_2,\vartheta_2}^{p_2,r_2,\tau_2}(\mathbb{R})$ to $CW_{\omega_3,\vartheta_3}^{p_3,r_3,\tau_3}(\mathbb{R})$.

$BM\big[CW\big(p_1, r_{1,}\omega_1, \vartheta_{1,}\tau_1; p_2, r_2, \omega_2, \vartheta_2\ , \tau_{2,}; p_3, r_3, \omega_3, \vartheta_3, \tau_3\big)\big]$ (shortly $BM\big[CW\big(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i\big)\big]$) denotes the space of all bilinear multipliers of type $CW\big(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i\big)$). We denote by

$$\|m\|_{CW(p_i,r_i,\omega_i,\vartheta_i,\tau_i)} = \|B_m\|.$$

In this work, we will assume that $\omega_1, \vartheta_1, \omega_2, \vartheta_2$ are weight functions of polynomial type.

Lemma 2.1 (Hölder Type Inequality for The Space $CW_{\omega,\vartheta}^{p,r,\tau}(\mathbb{R})$)

Assume that k_1 and k_2 are constant numbers such that $\omega_3 \approx k_1$, $\vartheta_3 \approx k_2$. If $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$, then there exists $C > 0$ such that

$$\|fg\|_{CW_{\omega_3,\vartheta_3}^{2,2,\tau_3}} \leq C\|f\|_{CW_{\omega_1,\vartheta_1}^{p_1,r_1,\tau_1}} \|g\|_{CW_{\omega_2,\vartheta_2}^{p_2,r_2,\tau_2}}$$

for all $f \in CW_{\omega_1,\vartheta_1}^{p_1,r_1,\tau_1}(\mathbb{R})$ and $g \in CW_{\omega_2,\vartheta_2}^{p_2,r_2,\tau_2}(\mathbb{R})$.

Proof. Let $f \in CW_{\omega_1,\vartheta_1}^{p_1,r_1,\tau_1}(\mathbb{R})$ and $g \in CW_{\omega_2,\vartheta_2}^{p_2,r_2,\tau_2}(\mathbb{R})$. Take any $h \in S(\mathbb{R})$. The following equation is known that

$$W_\tau(fg,h)(x,w) = \frac{1}{\sqrt{\tau(1-\tau)}} e^{\frac{2\pi i x w}{\tau}} V_{D_{\frac{\tau}{\tau-1}}h}^{\tau}(fg)(x,w), (0 < \tau < 1).$$

Then from the Hölder inequality, we get

$$\|fg\|_{CW_{\omega_3,\vartheta_3}^{2,2,\tau_3}} = \|fg\|_{2,\omega_3} + \|W_\tau(fg,h)\|_{2,\vartheta_3}$$

$$\approx \|fg\|_2 + \|W_\tau(fg,h)\|_2$$

$$\leq \|f\|_{p_1}\|g\|_{p_2} + \frac{1}{\sqrt{\tau(1-\tau)}} \left\| V_{D_{\frac{\tau}{\tau-1}}h}^{\tau}(fg) \right\|_2$$

$$= \|f\|_{p_1}\|g\|_{p_2} + \frac{1}{\sqrt{\tau(1-\tau)}} \left\| V_{D_{\frac{\tau}{\tau-1}}h}(fg)\left(\frac{x}{1-\tau}, \frac{w}{\tau}\right) \right\|_2$$

$$= \|f\|_{p_1}\|g\|_{p_2} + \left\| V_{D_{\frac{\tau}{\tau-1}}h}(fg) \right\|_2. \tag{5.1}$$

Moreover since the Gabor transform is an isometry from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}^2)$ (Gröchenig, 2001), using the inequality (5.1) we achieve

$$\|fg\|_{CW^{2,2,\tau_3}_{\omega_3,\vartheta_3}} \leq \|f\|_{p_1}\|g\|_{p_2} + \|fg\|_2 \left\|D_{\frac{\tau}{\tau-1}}h\right\|_2$$

$$\leq \|f\|_{p_1}\|g\|_{p_2} + \|f\|_{p_1}\|g\|_{p_2} \left\|D_{\frac{\tau}{\tau-1}}h\right\|_2$$

$$= \|f\|_{p_1}\|g\|_{p_2} + \|f\|_{p_1}\|g\|_{p_2}\|h\|_2$$

$$= \|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2} + \|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}\|h\|_2$$

$$\leq \{1+\|h\|_2\}\{\|f\|_{p_1,\omega_1} + \|W_\tau(f,h)\|_{r_1,\vartheta_1}\}\{\|g\|_{p_2,\omega_2} + \|W_\tau(f,h)\|_{r_2,\vartheta_2}\}$$

$$= C\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}, \text{ where } C = \{1+\|h\|_2\}. \qquad \blacksquare$$

Now let's give a theorem as an example of bilinear multipliers. In this work, the weight functions $\{\omega_1(\alpha z) + \vartheta_1((\alpha z, 0))\vartheta_1((\alpha z\tau_1, 0))\}$ and $\{\omega_2(\beta z) + \vartheta_2((\beta z, 0))\vartheta_2((\beta z\tau_2, 0))\}$ will be denoted by the symbols $u_\alpha(z)$ and $v_\beta(z)$ for $\alpha, \beta \in \mathbb{R}$, respectively. Since the weight functions $\omega_1, \vartheta_1, \omega_2, \vartheta_2$ are polynomial type, u_α and v_β are symmetric functions for $\alpha, \beta \in \mathbb{R}$. That means $u_\alpha(-z) = u_\alpha(z)$, $v_\beta(-z) = v_\beta(z)$ for $z \in \mathbb{R}$.

Theorem 2.2 Let $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$. Assume that k_1 and k_2 are constant numbers. If $\omega_3 \approx k_1$, $\vartheta_3 \approx k_2$ and $K \in L^1_\omega(\mathbb{R})$ such that $\omega(z) = u_1(z)v_1(z)$, then $m(\xi,\eta) = \widehat{K}(\xi-\eta)$ is a bilinear multiplier on $\mathbb{R}$ of type $CW(p_1, r_{1,}\omega_3, \vartheta_{1,}\tau_1; p_2, r_2, \omega_3, \vartheta_2\ , \tau_{2,}; 2, 2, \omega_3, \vartheta_3, \tau_3)$. Moreover there exists $C > 0$ such that

$$\|m\|_{CW(p_1,r_{1,}\omega_3,\vartheta_{1,}\tau_1;p_2,r_2,\omega_3,\vartheta_2\ ,\tau_{2,};2,2,\omega_3,\vartheta_3,\tau_3)} \leq C\|K\|_{1,\omega}.$$

Proof. Let any $f, g \in S(\mathbb{R})$ be given. It is known that $f(x-z) \in CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}(\mathbb{R})$ and $g(x+z) \in CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}(\mathbb{R})$ (Kulak & Ömerbeyoğlu, 2021:188-200). If we use Lemma 2.1 and the (2.6) equality in (Kulak & Gürkanlı, 2013), then we have

$$\|B_m\,(f,g)\|_{CW^{2,2,\tau_3}_{\omega_3,\vartheta_3}} = \left\|\int_{\mathbb{R}} f(x-z)g(x+z)K(z)dz\right\|_{CW^{2,2,\tau_3}_{\omega_3,\vartheta_3}}$$

$$\leq \int_{\mathbb{R}} \|f(x-z)g(x+z)\|_{CW^{2,2,\tau_3}_{\omega_3,\vartheta_3}} |K(z)|dz$$

$$\leq \int_{\mathbb{R}} C\|T_z f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}} \|T_{-z} g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} |K(z)|dz. \tag{5.2}$$

Furthermore from Theorem 2.8 in (Kulak & Ömerbeyoğlu, 2021:188-200), we can write

$$\|T_z f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}} \leq \{\omega_1(z) + \vartheta_1((z,0))\vartheta_1((z\tau_1,0))\}\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}$$

$$= u_1(z)\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}} \tag{5.3}$$

and

$$\|T_{-z} g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} \leq \{\omega_2(-z) + \vartheta_2((-z,0))\vartheta_2((-z\tau_2,0))\}\|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}$$

$$= v_1(z)\|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} \tag{5.4}$$

Combining the inequalities (5.2), (5.3) and (5.4), we get

$$\|B_m(f,g)\|_{CW^{2,2,\tau_3}_{\omega_3,\vartheta_3}} \leq C\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}} \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} \int_{\mathbb{R}} |K(z)|\omega(z)dz$$

$$= C\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}} \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} \|K\|_{1,\omega}. \tag{5.5}$$

So we say that $m(\xi,\eta) = \widehat{K}(\xi-\eta)$ is a bilinear multiplier. Also we obtain

$$\|m\|_{CW(p_1,r_1,\omega_3,\vartheta_1,\tau_1;p_2,r_2,\omega_3,\vartheta_2,\tau_2;2,2,\omega_3,\vartheta_3,\tau_3)}$$

$$= \sup_{\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}} \leq 1, \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} \leq 1} \frac{\|B_m(f,g)\|_{CW^{2,2,\tau_3}_{\omega_3,\vartheta_3}}}{\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}} \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}}$$

$$\leq C\|K\|_{1,\omega}.$$ ■

Definition 2.3 Let $1 \leq p_1, r_1, p_2, r_2 p_3, r_3 < \infty$, $\tau_1, \tau_{2,} \tau_3 \in (0,1)$ and $\omega_1, \vartheta_1, \omega_2, \vartheta_2\ \omega_3, \vartheta_3$ be weight functions. Suppose that $\omega_1, \vartheta_1, \omega_2, \vartheta_2$ are weight functions of polynomial type. We denote by

$$\widetilde{M}\left[CW\left(p_1, r_{1,} \omega_1, \vartheta_{1,} \tau_1; p_2, r_2, \omega_2, \vartheta_2\ , \tau_{2,}; p_3, r_3, \omega_3, \vartheta_3, \tau_3\right)\right]$$

(shortly $\widetilde{M}\left[CW\left(p_i, r_{i,} \omega_i, \vartheta_{i,} \tau_i\right)\right]$) the space of measurable functions $M: \mathbb{R} \to \mathbb{C}$ such that $m(\xi, \eta) = M(\xi - \eta) \in BM\left[CW\left(p_i, r_{i,} \omega_i, \vartheta_{i,} \tau_i\right)\right]$, that is to say

$$B_M(f, g)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi) \hat{g}\,(\eta) M(\xi - \eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta$$

extends to bounded bilinear map from $CW_{\omega_1, \vartheta_1}^{p_1, r_1, \tau_1}(\mathbb{R}) \times CW_{\omega_2, \vartheta_2}^{p_2, r_2, \tau_2}(\mathbb{R})$ to $CW_{\omega_3, \vartheta_3}^{p_3, r_3, \tau_3}(\mathbb{R})$. Also we denote

$$\|M\|_{CW(p_i, r_{i,} \omega_i, \vartheta_{i,} \tau_i)} = \|B_M\|$$

Theorem 2.4 Let $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$, $\omega_i(z) = (1 + |z|)^{a_i}$, $\vartheta_i(z, x) = (1 + |z| + |x|)^{b_i}$, $a_i, b_i \geq 0$ (i=1,2). Assume that k_1 and k_2 are constant numbers. If $\omega_3 \approx k_1$, $\vartheta_3 \approx k_2$, $\mu \in M(\omega)$, $\omega(z) = u_1(z) v_1(z)$, and $m(\xi, \eta) = \hat{\mu}(\alpha\xi + \beta\eta)$ for $\alpha, \beta \in \mathbb{R}$, then $m \in BM\left[CW\left(p_1, r_{1,} \omega_3, \vartheta_{1,} \tau_1; p_2, r_2, \omega_3, \vartheta_2\ , \tau_{2,}; 2, 2, \omega_3, \vartheta_3, \tau_3\right)\right]$. Furthermore there exists $C > 0$ such that

$$\|m\|_{CW(p_1, r_{1,} \omega_3, \vartheta_{1,} \tau_1; p_2, r_2, \omega_3, \vartheta_2\ , \tau_{2,}; 2, 2, \omega_3, \vartheta_3, \tau_3)} \leq C \|\mu\|_\omega$$

Proof. Take any $f, g \in S(\mathbb{R})$. From Theorem 2.3 in (Kulak & Gürkanlı, 2013), we know that

$$B_m(f, g)(x) = \int_{\mathbb{R}} f(x - \alpha z) g(x - \beta z) d\mu(z)\ . \qquad (5.6)$$

By from (Kulak & Ömerbeyoğlu, 2021:188-200), we can write

$$\|T_{\alpha z} f\|_{CW_{\omega_1, \vartheta_1}^{p_1, r_1, \tau_1}} \leq \left\{\omega_1(\alpha z) + \vartheta_1\left((\alpha z, 0)\right) \vartheta_1\left((\alpha z \tau_1, 0)\right)\right\} \|f\|_{CW_{\omega_1, \vartheta_1}^{p_1, r_1, \tau_1}}$$

(5.7)

and

$$\left\|T_{\beta z}g\right\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} \leq \{\omega_2(\beta z)+\vartheta_2\big((\beta z,0)\big)\vartheta_2\big((\beta z\tau_2,0)\big)\}\|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}$$

(5.8)

Then by using (5.6), (5.7), (5.8) and Lemma 2.1, we have

$$\|B_m\ (f,g)\|_{CW^{2,2,\tau_3}_{\omega_3,\vartheta_3}} \leq \int_{\mathbb{R}} \ \|f(x-\alpha z)g(x-\beta z)\|_{CW^{2,2,\tau_3}_{\omega_3,\vartheta_3}} d|\mu|(z)$$

$$\leq \int_{\mathbb{R}} \ C\|T_{\alpha z}f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\left\|T_{\beta z}g\right\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} d|\mu|(z)$$

$$\leq \int_{\mathbb{R}} \ Cu_\alpha(z)v_\beta(z)\ \|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\ \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} d|\mu|(z)$$

$$= C\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\ \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} \int_{\mathbb{R}} \ u_\alpha(z)v_\beta(z)\ d|\mu|(z). \tag{5.9}$$

Firstly, assume that $|\alpha| \leq 1, |\beta| \leq 1$. Then

$$u_\alpha(z) = \omega_1(\alpha z)+\vartheta_1\big((\alpha z,0)\big)\vartheta_1\big((\alpha z\tau_1,0)\big)$$

$$= (1+|\alpha z|)^{a_1} + (1+|\alpha z|)^{b_1}(1+|\alpha z\tau_1|)^{b_1}$$

$$\leq (1+|z|)^{a_1} + (1+|z|)^{b_1}(1+|\tau_1 z|)^{b_1} = u_1(z)$$

Similarly we have $v_\beta(z) \leq v_1(z)$. Hence by (5.9), we find

$$\|B_m\ (f,g)\|_{CW^{2,2,\tau_3}_{\omega_3,\vartheta_3}} \leq C\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\ \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} \int_{\mathbb{R}} \ u_1(z)v_1(z)\ d|\mu|(z)$$

$$= C\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\ \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}\|\mu\|_\omega. \tag{5.10}$$

So we get $m \in BM\big[CW\big(p_1,r_{1,}\omega_3,\vartheta_{1,}\tau_1;p_2,r_2,\omega_3,\vartheta_2\ ,\tau_{2,};2,2,\omega_3,\vartheta_3,\tau_3\big)\big]$ and from (5.10)

$$\|m\|_{CW(p_1,r_{1,}\omega_3,\vartheta_{1,}\tau_1;p_2,r_2,\omega_3,\vartheta_2\ ,\tau_{2,};2,2,\omega_3,\vartheta_3,\tau_3)}$$

$$= \sup_{\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}} \leq 1, \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} \leq 1} \frac{\|B_m\,(f,g)\|_{CW^{2,2,\tau_3}_{\omega_3,\vartheta_3}}}{\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}} \leq C\,\|\mu\|_\omega$$

Now, assume that $|\alpha| > 1, |\beta| > 1$. Then

$$u_\alpha(z) = \omega_1(\alpha z) + \vartheta_1\big((\alpha z, 0)\big)\vartheta_1\big((\alpha z\tau_1, 0)\big)$$

$$< (|\alpha| + |\alpha z|)^{a_1} + (|\alpha| + |\alpha z|)^{b_1}(|\alpha| + |\alpha\tau_1 z|)^{b_1}$$

$$= |\alpha|^{a_1}(1 + |z|)^{a_1} + |\alpha|^{2b_1}(1 + |z|)^{b_1}(1 + |\tau_1 z|)^{b_1}$$

$$< |\alpha|^{a_1}|\alpha|^{2b_1}(1 + |z|)^{a_1} + |\alpha|^{a_1}|\alpha|^{2b_1}(1 + |z|)^{b_1}(1 + |\tau_1 z|)^{b_1}$$

$$= |\alpha|^{a_1}|\alpha|^{2b_1}\big\{\omega_1(z) + \vartheta_1\big((z, 0)\big)\vartheta_1\big((z\tau_1, 0)\big)\big\}$$

$$= |\alpha|^{a_1+2b_1}u_1(z).$$

Also under same conditions, we have $v_\beta(z) < |\beta|^{a_2+2b_2}v_1(z)$. So by (5.9)

$$\|B_m\,(f,g)\|_{CW^{2,2,\tau_3}_{\omega_3,\vartheta_3}}$$

$$< C|\alpha|^{a_1+2b_1}|\beta|^{a_2+2b_2}\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\,\|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}\|\mu\|_\omega. \tag{5.11}$$

Thus we achieve

$$m \in BM\big[CW\big(p_1, r_{1,}\omega_3, \vartheta_{1,}\tau_1; p_2, r_2, \omega_3, \vartheta_2\,, \tau_{2,}; 2, 2, \omega_3, \vartheta_3, \tau_3\big)\big]$$

and from (5.11)

$$\|m\|_{CW(p_1,r_{1,}\omega_3,\vartheta_{1,}\tau_1;p_2,r_2,\omega_3,\vartheta_2\,,\tau_{2,};2,2,\omega_3,\vartheta_3,\tau_3)}$$

$$= \sup_{\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}} \leq 1, \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} \leq 1} \frac{\|B_m\,(f,g)\|_{CW^{2,2,\tau_3}_{\omega_3,\vartheta_3}}}{\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}}$$

$$< C|\alpha|^{a_1+2b_1}|\beta|^{a_2+2b_2}\,\|\mu\|_\omega.$$

Suppose that $|\alpha| > 1, |\beta| \leq 1$. Since $u_\alpha(z) < |\alpha|^{a_1+2b_1}u_1(z)$ and $v_\beta(z) \leq v_1(z)$, it is obtained that

$$\|B_m(f,g)\|_{CW^{2,2,\tau_3}_{\omega_3,\vartheta_3}} < C|\alpha|^{a_1+2b_1}\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}\|\mu\|_{\omega}. \quad (5.12)$$

Hence we find $m \in$ $BM\left[CW\left(p_1, r_{1,}\omega_3, \vartheta_{1,}\tau_1; p_2, r_2, \omega_3, \vartheta_2\ , \tau_{2,}; 2, 2, \omega_3, \vartheta_3, \tau_3\right)\right]$ and by (5.12)

$$\|m\|_{CW(p_1,r_{1,}\omega_3,\vartheta_{1,}\tau_1;p_2,r_2,\omega_3,\vartheta_2\ ,\tau_{2,};2,2,\omega_3,\vartheta_3,\tau_3)}$$

$$= \sup_{\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\leq 1, \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}\leq 1} \frac{\|B_m(f,g)\|_{CW^{2,2,\tau_3}_{\omega_3,\vartheta_3}}}{\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}} < C|\alpha|^{a_1+2b_1}\|\mu\|_{\omega}.$$

Finally suppose that $|\alpha| \leq 1, |\beta| > 1$. Again since $u_\alpha(z) \leq u_1(z)$ and $v_\beta(z) < |\beta|^{a_2+2b_2}v_1(z)$, we observe

$$\|B_m(f,g)\|_{CW^{2,2,\tau_3}_{\omega_3,\vartheta_3}} < C|\beta|^{a_2+2b_2}\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}\|\mu\|_{\omega}. \quad (5.13)$$

So we obtain $m \in$ $BM\left[CW\left(p_1, r_{1,}\omega_3, \vartheta_{1,}\tau_1; p_2, r_2, \omega_3, \vartheta_2\ , \tau_{2,}; 2, 2, \omega_3, \vartheta_3, \tau_3\right)\right]$ and by (5.13)

$$\|m\|_{CW(p_1,r_{1,}\omega_3,\vartheta_{1,}\tau_1;p_2,r_2,\omega_3,\vartheta_2\ ,\tau_{2,};2,2,\omega_3,\vartheta_3,\tau_3)}$$

$$= \sup_{\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\leq 1, \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}\leq 1} \frac{\|B_m(f,g)\|_{CW^{2,2,\tau_3}_{\omega_3,\vartheta_3}}}{\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}}$$

$$< C|\beta|^{a_2+2b_2}\|\mu\|_{\omega}.$$ ■

Theorem 2.5 Let $m \in BM\left[CW\left(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i\right)\right]$. Then

$$M_{(a,b)}m \in BM\left[CW\left(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i\right)\right]$$

for each $(a,b) \in \mathbb{R}^2$. Furthermore

$$\left\|M_{(a,b)}m\right\|_{CW(p_i,r_{i,}\omega_i,\vartheta_{i,}\tau_i)} \leq u_1(a)v_1(b)\|m\|_{CW(p_i,r_{i,}\omega_i,\vartheta_{i,}\tau_i)}$$

Proof. Let $f, g \in S(\mathbb{R})$ be given. It is known that

$$B_{M_{(a,b)}m}(f,g) = B_m(T_{-a}f, T_{-b}g) \quad (5.14)$$

by (Kulak & Gürkanlı, 2013). Moreover we have

$$\|T_{-a}f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}} \le \{\omega_1(-a) + \vartheta_1((-a,0))\vartheta_1((-a\tau_2,0))\}\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}$$

$$= u_1(a)\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}} \tag{5.15}$$

and

$$\|T_{-b}g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} \le \{\omega_2(-b) + \vartheta_2((-b,0))\vartheta_2((-b\tau_2,0))\}\|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}$$

$$= v_1(b)\|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}\ . \tag{5.16}$$

If we use assumption $m \in BM[CW(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i)]$ and combine (5.14), (5.15) and (5.16), then we get

$$\left\|B_{M_{(a,b)}m}(f,g)\right\|_{CW^{p_3,r_3,\tau_3}_{\omega_3,\vartheta_3}} = \|B_m(T_{-a}f, T_{-b}g)\|_{CW^{p_3,r_3,\tau_3}_{\omega_3,\vartheta_3}}$$

$$\le \|B_m\|\|T_{-a}f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\ \|T_{-b}g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}.$$

$$\le u_1(a)v_1(b)\|B_m\|\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\ \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} \tag{5.17}$$

and so $M_{(a,b)}m \in BM[CW(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i)]$. Hence by (5.17), we conclude

$$\left\|M_{(a,b)}m\right\|_{CW(p_i,r_{i,}\omega_i,\vartheta_{i,}\tau_i)}$$

$$= \sup_{\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}} \le 1, \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} \le 1} \frac{\left\|B_{M_{(a,b)}m}(f,g)\right\|_{CW^{p_3,r_3,\tau_3}_{\omega_3,\vartheta_3}(\mathbb{R})}}{\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}}$$

$$\le u_1(a)v_1(b)\|m\|_{CW(p_i,r_{i,}\omega_i,\vartheta_{i,}\tau_i)}. \qquad \blacksquare$$

Lemma 2.6 Suppose that ω, ϑ are polynomial type weight functions. Let $f \in CW^{p,r,\tau}_{\omega,\vartheta}(\mathbb{R})$. Then $D_s f \in CW^{p,r,\tau}_{\omega,\vartheta}(\mathbb{R})$ for each $0 \ne s \in \mathbb{R}$. Moreover

$\|D_s f\|_{CW^{p,r,\tau}_{\omega,\vartheta}} \le C|s|^{-\frac{b}{r}}\|f\|_{CW^{p,r,\tau}_{\omega,\vartheta}}$, if $|s| \le 1$

$$\|D_s f\|_{CW^{p,r,\tau}_{\omega,\vartheta}} < C|s|^{\frac{1}{2}+\frac{a}{p}+\frac{b}{r}}\|f\|_{CW^{p,r,\tau}_{\omega,\vartheta}}, \text{ if } |s| > 1, \text{ for some } C > 0.$$

Proof. Take any $f \in CW^{p,r,\tau}_{\omega,\vartheta}(\mathbb{R})$. If we set $\frac{t}{s} = u$, then

$$\|D_s f\|_{p,\omega} = |s|^{-\frac{1}{2}}\left\{\int\limits_{\mathbb{R}} \left|f\left(\frac{t}{s}\right)\right|^p \omega(t)dt\right\}^{\frac{1}{p}}$$

$$= |s|^{-\frac{1}{2}}\left\{\int_{\mathbb{R}} |f(u)|^p\, \omega(su)sdu\right\}^{\frac{1}{p}}$$

$$= |s|^{\frac{1}{2}}\left\{\int_{\mathbb{R}} |f(u)|^p\, (1+|su|)^a du\right\}^{\frac{1}{p}} \tag{5.18}$$

Let $g \in S(\mathbb{R})$ be given. Then $D_{s^{-1}}g,\ D_{\frac{\tau}{\tau-1}}g \in S(\mathbb{R})$ and so $D_{\frac{\tau}{\tau-1}}(D_{s^{-1}}g) \in S(\mathbb{R})$. Also since different windows yield equivalent norms for a Gabor transform by Proposition 11.3.2 in (Gröchenig, 2001), there exists $C_1 > 0$ such that

$$\|W_\tau(D_s f, g)\|_{r,\vartheta} = \left\|W_\tau(f, D_{s^{-1}}g)\left(\frac{x}{s}, sw\right)\right\|_{r,\vartheta}$$

$$= \left\|\frac{1}{\sqrt{\tau(1-\tau)}}e^{\frac{2\pi ixw}{\tau}}V^\tau_{D_{\frac{\tau}{\tau-1}}(D_{s^{-1}}g)}f\left(\frac{x}{s}, sw\right)\right\|_{r,\vartheta}$$

$$= \frac{1}{\sqrt{\tau(1-\tau)}}\left\|V_{D_{\frac{\tau}{\tau-1}}(D_{s^{-1}}g)}f\left(\frac{x}{s(1-\tau)}, \frac{sw}{\tau}\right)\right\|_{r,\vartheta}$$

$$\leq C_1\frac{1}{\sqrt{\tau(1-\tau)}}\left\|V_{D_{\frac{\tau}{\tau-1}}g}f\left(\frac{x}{s(1-\tau)}, \frac{sw}{\tau}\right)\right\|_{r,\vartheta}$$

$$= C_1\left\|\frac{1}{\sqrt{\tau(1-\tau)}}e^{\frac{2\pi ixw}{\tau}}V_{D_{\frac{\tau}{\tau-1}}g}f\left(\frac{x}{s(1-\tau)}, \frac{sw}{\tau}\right)\right\|_{r,\vartheta}$$

$$= C_1\left\|\frac{1}{\sqrt{\tau(1-\tau)}}e^{\frac{2\pi ixw}{\tau}}V^\tau_{D_{\frac{\tau}{\tau-1}}g}f\left(\frac{x}{s}, sw\right)\right\|_{r,\vartheta}$$

$$= C_1 \left\| W_\tau(f,g)\left(\frac{x}{s}, sw\right) \right\|_{r,\vartheta}.$$

If we take $\frac{x}{s} = u$ and $sw = v$, then

$$\|W_\tau(D_s f,g)\|_{r,\vartheta} \le C_1 \left\| W_\tau(f,g)\left(\frac{x}{s}, sw\right) \right\|_{r,\vartheta}$$

$$= C_1 \left\{ \int_{\mathbb{R}} \left| W_\tau(f,g)\left(\frac{x}{s}, sw\right) \right|^r \vartheta(x,w)\,dxdw \right\}^{\frac{1}{r}}$$

$$= C_1 \left\{ \int_{\mathbb{R}} |W_\tau(f,g)(u,v)|^r \left(1 + |su| + \left|\frac{v}{s}\right|\right)^b dudv \right\}^{\frac{1}{r}}. \tag{5.19}$$

Assume that $|s| \le 1$. Hence by (5.18)

$$\|D_s f\|_{p,\omega} = |s|^{\frac{1}{2}} \left\{ \int_{\mathbb{R}} |f(u)|^p \, (1 + |su|)^a du \right\}^{\frac{1}{p}}$$

$$\le |s|^{\frac{1}{2}} \left\{ \int_{\mathbb{R}} |f(u)|^p \, (1 + |u|)^a du \right\}^{\frac{1}{p}} = \|f\|_{p,\omega} < \infty. \tag{5.20}$$

Since $f \in CW^{p,r,\tau}_{\omega,\vartheta}(\mathbb{R})$, we have $W_\tau(f,g) \in L^r_\vartheta(\mathbb{R}^2)$. So by (5.19)

$$\|W_\tau(D_s f,g)\|_{r,\vartheta} \le C_1 \left\{ \int_{\mathbb{R}} |W_\tau(f,g)(u,v)|^r \left(1 + |su| + \left|\frac{v}{s}\right|\right)^b dudv \right\}^{\frac{1}{r}}$$

$$\le C_1 \left\{ \int_{\mathbb{R}} |W_\tau(f,g)(u,v)|^r \left(1 + |u| + \left|\frac{v}{s}\right|\right)^b dudv \right\}^{\frac{1}{r}}$$

$$\le C_1 \left\{ \int_{\mathbb{R}} |W_\tau(f,g)(u,v)|^r \left(\frac{1}{|s|} + \left|\frac{u}{s}\right| + \left|\frac{v}{s}\right|\right)^b dudv \right\}^{\frac{1}{r}}$$

$$= C_1 \frac{1}{|s|}^{\frac{b}{r}} \left\{ \int_{\mathbb{R}} |W_\tau(f,g)(u,v)|^r \, (1+|u|+|v|)^b dudv \right\}^{\frac{1}{r}}$$

$$= C_1 \frac{1}{|s|}^{\frac{b}{r}} \|W_\tau(f,g)\|_{r,\vartheta} < \infty. \tag{5.21}$$

Combining (5.20) and (5.21), we achieve

$$\|D_s f\|_{CW^{p,r,\tau}_{\omega,\vartheta}} = \|D_s f\|_{p,\omega} + \|W_\tau(D_s f,g)\|_{r,\vartheta}$$

$$\leq \|f\|_{p,\omega} + C_1 \frac{1}{|s|}^{\frac{b}{r}} \|W_\tau(f,g)\|_{r,\vartheta}$$

$$\leq C \frac{1}{|s|}^{\frac{b}{r}} \left\{ \|f\|_{p,\omega} + \|W_\tau(f,g)\|_{r,\vartheta} \right\}$$

$=C|s|^{-\frac{b}{r}} \|f\|_{CW^{p,r,\tau}_{\omega,\vartheta}}$, where $C = max\{1, C_1\}$.

Now let $|s| > 1$. Then by (5.19)

$$\|D_s f\|_{p,\omega} = |s|^{\frac{1}{2}} \left\{ \int_{\mathbb{R}} |f(u)|^p \, (1+|su|)^a du \right\}^{\frac{1}{p}}$$

$$< |s|^{\frac{1}{2}} \left\{ \int_{\mathbb{R}} |f(u)|^p \, (|s|+|su|)^a du \right\}^{\frac{1}{p}}$$

$$= |s|^{\frac{1}{2}+\frac{a}{p}} \left\{ \int_{\mathbb{R}} |f(u)|^p \, (1+|u|)^a du \right\}^{\frac{1}{p}} = |s|^{\frac{1}{2}+\frac{a}{p}} \|f\|_{p,\omega} < \infty. \tag{5.22}$$

Again since $f \in CW^{p,r,\tau}_{\omega,\vartheta}(\mathbb{R})$, we have $W_\tau(f,g) \in L^r_\vartheta(\mathbb{R}^2)$. From (5.19)

$$\|W_\tau(D_s f,g)\|_{r,\vartheta} \leq C_1 \left\{ \int_{\mathbb{R}} |W_\tau(f,g)(u,v)|^r \left(1+|su|+\left|\frac{v}{s}\right|\right)^b dudv \right\}^{\frac{1}{r}}$$

$$< C_1 \left\{ \int_{\mathbb{R}} |W_\tau(f,g)(u,v)|^r \left(|s| + |su| + \left|\frac{v}{s}\right|\right)^b dudv \right\}^{\frac{1}{r}}$$

$$< C_1 \left\{ \int_{\mathbb{R}} |W_\tau(f,g)(u,v)|^r \left(|s| + |su| + |sv|\right)^b dudv \right\}^{\frac{1}{r}}$$

$$= C_1 |s|^{\frac{b}{r}} \left\{ \int_{\mathbb{R}} |W_\tau(f,g)(u,v)|^r \left(1 + |u| + |v|\right)^b dudv \right\}^{\frac{1}{r}}$$

$$= C_1 |s|^{\frac{b}{r}} \|W_\tau(f,g)\|_{r,\vartheta}. \tag{5.23}$$

By using (5.22) and (5.23), we obtain

$$\|D_s f\|_{CW^{p,r,\tau}_{\omega,\vartheta}} = \|D_s f\|_{p,\omega} + \|W_\tau(D_s f, g)\|_{r,\vartheta}$$

$$< |s|^{\frac{1}{2}+\frac{a}{p}} \|f\|_{p,\omega} + C_1 |s|^{\frac{b}{r}} \|W_\tau(f,g)\|_{r,\vartheta}$$

$$< |s|^{\frac{1}{2}+\frac{a}{p}+\frac{b}{r}} \|f\|_{p,\omega} + C_1 |s|^{\frac{1}{2}+\frac{a}{p}+\frac{b}{r}} \|W_\tau(f,g)\|_{r,\vartheta}$$

$$= C|s|^{\frac{1}{2}+\frac{a}{p}+\frac{b}{r}} \left\{ \|f\|_{p,\omega} + \|W_\tau(f,g)\|_{r,\vartheta} \right\}$$

$$= C|s|^{\frac{1}{2}+\frac{a}{p}+\frac{b}{r}} \|f\|_{CW^{p,r,\tau}_{\omega,\vartheta}}, \text{ where } C = max\{1, C_1\}.$$ ■

Theorem 2.7 Assume that ω_3 and ϑ_3 are polynomial type weight functions. Let $0 < s < \infty$ and $m \in BM\left[CW\left(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i\right)\right]$. Then

$$D_s m \in BM\left[CW\left(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i\right)\right]$$

Furthermore

$$\|D_s m\|_{CW(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i)} \leq Cs^{-\left(1+\frac{b_1}{r_1}+\frac{b_2}{r_2}+\frac{a_3}{p_3}+\frac{b_3}{r_3}\right)} \|m\|_{CW(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i)}, \text{ if } s \leq 1,$$

$$\|D_s m\|_{CW(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i)} < Cs^{\left(\frac{1}{2}+\frac{a_1}{p_1}+\frac{b_1}{r_1}+\frac{a_2}{p_2}+\frac{b_2}{r_2}+\frac{b_3}{r_3}\right)} \|m\|_{CW(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i)}, \text{ if } s > 1$$

for some $C > 0$.

Proof. Take any $f, g \in S(\mathbb{R})$. For $x \in \mathbb{R}$, we take $\frac{\xi}{s} = u, \frac{\eta}{s} = v$

$$B_{D_s m}(f,g)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi)\hat{g}\,(\eta) D_s m(\xi,\eta) e^{2\pi i\langle \xi+\eta, x\rangle} d\xi d\eta$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi)\hat{g}\,(\eta) s^{-1} m\left(\frac{\xi}{s}, \frac{\eta}{s}\right) e^{2\pi i\langle \xi+\eta, x\rangle} d\xi d\eta$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(su)\hat{g}\,(sv) s^{-1} m(u,v) e^{2\pi i\langle u+v, sx\rangle} s^2 du dv$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} s^{\frac{1}{2}} \hat{f}(su) s^{\frac{1}{2}} \hat{g}(sv) m(u,v) e^{2\pi i\langle u+v, sx\rangle} du dv$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} D_{s^{-1}} \hat{f}(u) D_{s^{-1}} \hat{g}(v) m(u,v) e^{2\pi i\langle u+v, sx\rangle} du dv$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{D_s f}(u)\, \widehat{D_s g}(v) m(u,v) e^{2\pi i\langle u+v, sx\rangle} du dv$$

$$= B_m(D_s f, D_s g)(sx) = s^{-\frac{1}{2}} D_{s^{-1}} B_m(D_s f, D_s g)(x) \tag{5.24}$$

Let $s \leq 1$. By Lemma 2.6, assumption $m \in BM\left[CW\left(p_i, r_{i,} \omega_i, \vartheta_{i,} \tau_i\right)\right]$ and equation (5.24), we find

$$\left\|B_{D_s m}(f,g)\right\|_{CW^{p_3,r_3,\tau_3}_{\omega_3,\vartheta_3}} = \left\|s^{-\frac{1}{2}} D_{s^{-1}} B_m(D_s f, D_s g)\right\|_{CW^{p_3,r_3,\tau_3}_{\omega_3,\vartheta_3}}$$

$$\leq C s^{-\left(1+\frac{a_3}{p_3}+\frac{b_3}{r_3}\right)} \|B_m\| \|D_s f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}} \|D_s g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}$$

$$\leq C s^{-\left(1+\frac{a_3}{p_3}+\frac{b_3}{r_3}\right)} \|B_m\| s^{-\frac{b_1}{r_1}} \|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}} s^{-\frac{b_2}{r_2}} \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}$$

$$= Cs^{-\left(1+\frac{b_1}{r_1}+\frac{b_2}{r_2}+\frac{a_3}{p_3}+\frac{b_3}{r_3}\right)}\|B_m\|\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}$$

for some $C > 0$. Thus we achieve

$$\|D_s m\|_{CW(p_i,r_{i,}\omega_i,\vartheta_{i,}\tau_i)} \le Cs^{-\left(1+\frac{b_1}{r_1}+\frac{b_2}{r_2}+\frac{a_3}{p_3}+\frac{b_3}{r_3}\right)}\|m\|_{CW(p_i,r_{i,}\omega_i,\vartheta_{i,}\tau_i)}.$$

Let $s > 1$. Again by Lemma 2.6, assumption $m \in BM\left[CW\left(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i\right)\right]$

and equation (5.24), we have

$$\left\|B_{D_s m}(f,g)\right\|_{CW^{p_3,r_3,\tau_3}_{\omega_3,\vartheta_3}} = \left\|s^{-\frac{1}{2}}D_{s^{-1}}B_m(D_s f, D_s g)\right\|_{CW^{p_3,r_3,\tau_3}_{\omega_3,\vartheta_3}}$$

$$< Cs^{-\frac{1}{2}}s^{\frac{b_3}{r_3}}\|B_m\|\|D_s f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\|D_s g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}$$

$$< Cs^{-\frac{1}{2}}s^{\frac{b_3}{r_3}}\|B_m\|s^{\left(\frac{1}{2}+\frac{a_1}{p_1}+\frac{b_1}{r_1}\right)}\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}s^{\left(\frac{1}{2}+\frac{a_2}{p_2}+\frac{b_2}{r_2}\right)}\|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}$$

$$= Cs^{\left(\frac{1}{2}+\frac{a_1}{p_1}+\frac{b_1}{r_1}+\frac{a_2}{p_2}+\frac{b_2}{r_2}+\frac{b_3}{r_3}\right)}\|B_m\|\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}$$

for some $C > 0$. Therefore we obtain

$$\|D_s m\|_{CW(p_i,r_{i,}\omega_i,\vartheta_{i,}\tau_i)} < Cs^{\left(\frac{1}{2}+\frac{a_1}{p_1}+\frac{b_1}{r_1}+\frac{a_2}{p_2}+\frac{b_2}{r_2}+\frac{b_3}{r_3}\right)}\|m\|_{CW(p_i,r_{i,}\omega_i,\vartheta_{i,}\tau_i)}. \qquad \blacksquare$$

Theorem 2.8 Let $m(s\xi, s\eta) = m(\xi,\eta)$, $0 < s < \infty$ and let ω_3, ϑ_3 be polynomial type weight functions. Then $m \in BM\left[CW\left(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i\right)\right]$ if and only if $D_s m \in BM\left[CW\left(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i\right)\right]$.

Proof. Let $f, g \in S(\mathbb{R})$ be given. Suppose that $s \neq 1$. Using the (5.24) and assumption $m(s\xi, s\eta) = m(\xi,\eta)$, we take $su = \xi$, $sv = \eta$

$$B_{D_s m}(f,g)(x) = s^{-\frac{1}{2}}D_{s^{-1}}B_m(D_s f, D_s g)(x)$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{D_s f}(u)\,\widehat{D_s g}(v)m(u,v)e^{2\pi i\langle u+v,sx\rangle}dudv$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} D_{s^{-1}} \hat{f}(u) D_{s^{-1}} \hat{g}(v) m(u,v) e^{2\pi i \langle u+v, sx \rangle} du dv$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} s \hat{f}(su) \hat{g}(sv) m(u,v) e^{2\pi i \langle u+v, sx \rangle} du dv$$

$$= s^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi) \hat{g}(\eta) m\left(\frac{\xi}{s}, \frac{\eta}{s}\right) e^{2\pi i \langle \xi+\eta, x \rangle} d\xi d\eta$$

$$= s^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi) \hat{g}(\eta) m(\xi,\eta) e^{2\pi i \langle \xi+\eta, x \rangle} d\xi d\eta$$

$$= s^{-1} B_m(f,g)(x)$$

Let $m \in BM\left[CW\left(p_i, r_{i,} \omega_i, \vartheta_{i,} \tau_i\right)\right]$. So we find

$$\left\|B_{D_s m}(f,g)\right\|_{CW^{p_3,r_3,\tau_3}_{\omega_3,\vartheta_3}} = \left\|s^{-1} B_m(f,g)\right\|_{CW^{p_3,r_3,\tau_3}_{\omega_3,\vartheta_3}}$$

$$\leq s^{-1} \|m\|_{CW(p_i,r_{i,}\omega_i,\vartheta_{i,}\tau_i)} \|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}} \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}.$$

Hence we achieve $D_s m \in BM\left[CW\left(p_i, r_{i,} \omega_i, \vartheta_{i,} \tau_i\right)\right]$. Suppose that

$$D_s m \in BM\left[CW\left(p_i, r_{i,} \omega_i, \vartheta_{i,} \tau_i\right)\right]$$

Then we have

$$\|B_m(f,g)\|_{CW^{p_3,r_3,\tau_3}_{\omega_3,\vartheta_3}} = \left\|s B_{D_s m}(f,g)\right\|_{CW^{p_3,r_3,\tau_3}_{\omega_3,\vartheta_3}}$$

$$\leq s \|D_s m\|_{CW(p_i,r_{i,}\omega_i,\vartheta_{i,}\tau_i)} \|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}} \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}.$$

Thus we obtain $m \in BM\left[CW\left(p_i, r_{i,} \omega_i, \vartheta_{i,} \tau_i\right)\right]$. ■

Theorem 2.9 Let $m \in BM\left[CW\left(p_i, r_{i,} \omega_i, \vartheta_{i,} \tau_i\right)\right]$. If $\Phi \in L^1_\omega(\mathbb{R}^2)$ such that $\omega(a,b) = u_1(a) v_1(b)$, then $\widehat{\Phi} m \in BM\left[CW\left(p_i, r_{i,} \omega_i, \vartheta_{i,} \tau_i\right)\right]$ and

$$\left\|\widehat{\Phi}m\right\|_{CW(p_i,r_i,\omega_i,\vartheta_i,\tau_i)} < \|\Phi\|_{1,\omega}\|m\|_{CW(p_i,r_i,\omega_i,\vartheta_i,\tau_i)}$$

Proof. Assume that $\Phi \in L^1_\omega(\mathbb{R}^2)$. Take any $f, g \in S(\mathbb{R})$. It is written by (Kulak &Gürkanlı, 2013)

$$B_{\widehat{\Phi}m}(f,g)(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \Phi(a,b)B_{M_{(-a,-b)}m}(f,g)\,(x)dadb$$

By assumption $m \in BM\left[CW\left(p_i, r_i, \omega_i, \vartheta_i, \tau_i\right)\right]$ and Theorem 2.5, we get $M_{(-a,-b)}m \in BM\left[CW\left(p_i, r_i, \omega_i, \vartheta_i, \tau_i\right)\right]$ and

$$\left\|M_{(a,b)}m\right\|_{CW(p_i,r_i,\omega_i,\vartheta_i,\tau_i)} \le u_1(a)v_1(b)\|m\|_{CW(p_i,r_i,\omega_i,\vartheta_i,\tau_i)}$$

So,

$$\|B_{\widehat{\Phi}m}(f,g)\|_{CW^{p_3,r_3,\tau_3}_{\omega_3,\vartheta_3}}$$

$$\le \int_{\mathbb{R}}\int_{\mathbb{R}} \left\|\Phi(a,b)B_{M_{(-a,-b)}m}(f,g)\right\|_{CW(p_i,r_i,\omega_i,\vartheta_i,\tau_i)} dadb$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} |\Phi(a,b)| \left\|B_{M_{(-a,-b)}m}(f,g)\right\|_{CW(p_i,r_i,\omega_i,\vartheta_i,\tau_i)} dadb$$

$$\le \int_{\mathbb{R}}\int_{\mathbb{R}} |\Phi(a,b)|\left\|M_{(a,b)}m\right\|_{CW(p_i,r_i,\omega_i,\vartheta_i,\tau_i)}\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} dadb$$

$$\le \int_{\mathbb{R}}\int_{\mathbb{R}} |\Phi(a,b)|u_1(a)v_1(b)\|m\|_{CW(p_i,r_i,\omega_i,\vartheta_i,\tau_i)}\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} dadb$$

$$= \|\Phi\|_{1,\omega}\|m\|_{CW(p_i,r_i,\omega_i,\vartheta_i,\tau_i)}\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}. \tag{5.25}$$

Therefore by (5.25), we observe that $\widehat{\Phi}m \in BM\left[CW\left(p_i, r_i, \omega_i, \vartheta_i, \tau_i\right)\right]$ and

$$\left\|\widehat{\Phi}m\right\|_{CW(p_i,r_i,\omega_i,\vartheta_i,\tau_i)} < \|\Phi\|_{1,\omega}\|m\|_{CW(p_i,r_i,\omega_i,\vartheta_i,\tau_i)}. \qquad \blacksquare$$

Theorem 2.10 Let $m \in BM\left[CW\left(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i\right)\right]$ and let ω_3, ϑ_3 be polynomial type weight functions.

a) If $\Phi \in L^1\left(\mathbb{R}^+, s^{-\left(1+\frac{a_1}{p_1}+\frac{b_1}{r_1}+\frac{a_2}{p_2}+\frac{b_2}{r_2}+\frac{b_3}{r_3}\right)}\right)$ such that $s \leq 1$, then

$$m_\Phi(\xi,\eta) = \int_0^\infty m(s\xi, s\eta)\Phi(s)\mathrm{ds} \in BM\left[CW\left(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i\right)\right]$$

Furthermore, $\|m_\Phi\|_{CW(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i)}$

$\leq C\|\Phi\|_{L^1\left(\mathbb{R}^+, s^{-\left(1+\frac{a_1}{p_1}+\frac{b_1}{r_1}+\frac{a_2}{p_2}+\frac{b_2}{r_2}+\frac{b_3}{r_3}\right)}\right)}\|m\|_{CW(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i)}$ for some $C > 0$.

b) If $\Phi \in L^1\left(\mathbb{R}^+, s^{\left(\frac{1}{2}+\frac{b_1}{r_1}+\frac{b_2}{r_2}+\frac{a_3}{p_3}+\frac{b_3}{r_3}\right)}\right)$ such that $s > 1$, then

$$m_\Phi(\xi,\eta) = \int_0^\infty m(s\xi, s\eta)\Phi(s)\mathrm{ds} \in BM\left[CW\left(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i\right)\right]$$

Moreover $\|m_\Phi\|_{CW(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i)}$

$\leq C\|\Phi\|_{L^1\left(\mathbb{R}^+, s^{\left(\frac{1}{2}+\frac{b_1}{r_1}+\frac{b_2}{r_2}+\frac{a_3}{p_3}+\frac{b_3}{r_3}\right)}\right)}\|m\|_{CW(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i)}$ for some $C > 0$.

Proof. Let us take $f, g \in S(\mathbb{R})$. Then by Fubini Theorem,

$$B_{m_\Phi}(f,g)(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)m_\Phi(\xi,\eta)e^{2\pi i\langle \xi+\eta, x\rangle}d\xi d\eta$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)\left\{\int_0^\infty m(s\xi, s\eta)\Phi(s)\mathrm{ds}\right\}e^{2\pi i\langle \xi+\eta, x\rangle}d\xi d\eta$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)\left\{\int_0^\infty D_{s^{-1}}m(\xi,\eta)\Phi(s)s^{-\frac{1}{2}}\mathrm{ds}\right\}e^{2\pi i\langle \xi+\eta, x\rangle}d\xi d\eta$$

$$= \int_0^\infty B_{D_{s^{-1}}m}(f,g)(x)\Phi(s)s^{-\frac{1}{2}}\mathrm{ds}. \tag{5.26}$$

a) Let $s \leq 1$. If we use $m \in BM\left[CW\left(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i\right)\right]$ and the equality (5.26), then we have $D_{s^{-1}}m \in BM\left[CW\left(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i\right)\right]$ and

$$\left\|B_{m_\Phi}(f,g)\right\|_{CW^{p_3,r_3,\tau_3}_{\omega_3,\vartheta_3}} \leq \int_0^\infty \left\|B_{D_{s^{-1}}m}(f,g)\right\| |\Phi(s)| s^{-\frac{1}{2}} ds$$

$$\leq \int_0^\infty \|D_{s^{-1}}m\|_{CW(p_i,r_{i,}\omega_i,\vartheta_{i,}\tau_i)} \|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}} \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} |\Phi(s)| s^{-\frac{1}{2}} ds$$

$$\leq \int_0^\infty Cs^{-\left(1+\frac{a_1}{p_1}+\frac{b_1}{r_1}+\frac{a_2}{p_2}+\frac{b_2}{r_2}+\frac{b_3}{r_3}\right)} \|m\|_{CW(p_i,r_{i,}\omega_i,\vartheta_{i,}\tau_i)} \|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}} \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} |\Phi(s)| ds$$

$$= C\|m\|_{CW(p_i,r_{i,}\omega_i,\vartheta_{i,}\tau_i)} \|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}} \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} \int_0^\infty s^{-\left(1+\frac{a_1}{p_1}+\frac{b_1}{r_1}+\frac{a_2}{p_2}+\frac{b_2}{r_2}+\frac{b_3}{r_3}\right)} |\Phi(s)| ds$$

$$= C\|m\|_{CW(p_i,r_{i,}\omega_i,\vartheta_{i,}\tau_i)} \|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}} \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} \|\Phi\|_{L^1\left(\mathbb{R}^+, s^{-\left(1+\frac{a_1}{p_1}+\frac{b_1}{r_1}+\frac{a_2}{p_2}+\frac{b_2}{r_2}+\frac{b_3}{r_3}\right)}\right)}$$

and so $m_\Phi \in BM\left[CW\left(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i\right)\right]$. Then

$$\|m_\Phi\|_{CW(p_i,r_{i,}\omega_i,\vartheta_{i,}\tau_i)} \leq C\|\Phi\|_{L^1\left(\mathbb{R}^+, s^{-\left(1+\frac{a_1}{p_1}+\frac{b_1}{r_1}+\frac{a_2}{p_2}+\frac{b_2}{r_2}+\frac{b_3}{r_3}\right)}\right)} \|m\|_{CW(p_i,r_{i,}\omega_i,\vartheta_{i,}\tau_i)}$$

b) Assume that $s > 1$. Then by $m \in BM\left[CW\left(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i\right)\right]$ and the equality (5.26), we find $D_{s^{-1}}m \in BM\left[CW\left(p_i, r_{i,}\omega_i, \vartheta_{i,}\tau_i\right)\right]$ and

$$\left\|B_{m_\Phi}(f,g)\right\|_{CW^{p_3,r_3,\tau_3}_{\omega_3,\vartheta_3}} \leq \int_0^\infty \left\|B_{D_{s^{-1}}m}(f,g)\right\| |\Phi(s)| s^{-\frac{1}{2}} ds$$

$$< \int_0^\infty \|D_{s^{-1}}m\|_{CW(p_i,r_{i,}\omega_i,\vartheta_{i,}\tau_i)} \|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}} \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} |\Phi(s)| s^{-\frac{1}{2}} ds$$

$$< \int_0^\infty Cs^{\left(\frac{1}{2}+\frac{b_1}{r_1}+\frac{b_2}{r_2}+\frac{a_3}{p_3}+\frac{b_3}{r_3}\right)} \|m\|_{CW(p_i,r_{i,}\omega_i,\vartheta_{i,}\tau_i)} \|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}} \|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}} |\Phi(s)| ds$$

$$= C\|m\|_{CW(p_i,r_i,\omega_i,\vartheta_i,\tau_i)}\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}\int_0^\infty s^{\left(\frac{1}{2}+\frac{b_1}{r_1}+\frac{b_2}{r_2}+\frac{a_3}{p_3}+\frac{b_3}{r_3}\right)}|\Phi(s)|\mathrm{ds}$$

$$= C\|m\|_{CW(p_i,r_i,\omega_i,\vartheta_i,\tau_i)}\|f\|_{CW^{p_1,r_1,\tau_1}_{\omega_1,\vartheta_1}}\|g\|_{CW^{p_2,r_2,\tau_2}_{\omega_2,\vartheta_2}}\|\Phi\|_{L^1\left(\mathbb{R}^+,s^{\left(\frac{1}{2}+\frac{b_1}{r_1}+\frac{b_2}{r_2}+\frac{a_3}{p_3}+\frac{b_3}{r_3}\right)}\right)}.$$

Thus we obtain $m_\Phi \in BM\left[CW(p_i,r_i,\omega_i,\vartheta_i,\tau_i)\right]$ and

$$\|m_\Phi\|_{CW(p_i,r_i,\omega_i,\vartheta_i,\tau_i)} \leq C\|\Phi\|_{L^1\left(\mathbb{R}^+,s^{\left(\frac{1}{2}+\frac{b_1}{r_1}+\frac{b_2}{r_2}+\frac{a_3}{p_3}+\frac{b_3}{r_3}\right)}\right)}\|m\|_{CW(p_i,r_i,\omega_i,\vartheta_i,\tau_i)}\ . \blacksquare$$

3. CONCLUSION

Theory of bilinear multipliers has been studied in a number of papers (Coifman & Meyer, 1978; Gilbert & Nahmod, 2000; Gilbert & Nahmod, 2001; Grafakos & Kalton, 2001). In our previous works, we investigated bilinear multipliers and gave examples for weighted Lebesgue spaces, small Lebesgue spaces, weighted Wiener amalgam spaces, weighted Lorentz spaces, variable exponenet Lebesgue spaces, variable exponent Wiener amalgam spaces, variable exponent Lorentz spaces, etc (Kulak & Gürkanlı, 2013; Kulak & Gürkanlı, 2014; Kulak & Gürkanlı, 2017; Kulak & Gürkanlı, 2021). This chapter deals with the theory of bilinear multipliers on $CW^{p,r,\tau}_{\omega,\vartheta}(\mathbb{R})$ which was studied by (Kulak & Ömerbeyoğlu, 2021). In this work, the bilinear multipliers space is defined, and then exemplary theorems are proved for this function space characterized by the Wigner transform.

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Chapter 6

ON ALMOST KENMOTSU MANIFOLDS ADMITTING NULLITY DISTRIBUTIONS

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1. INTRODUCTION

Recently the field of nullity distributions has become very interesting topic in differential geometry. Gray (Gray, 1966) and Tanno (Tanno, 1978) introduced the notion of κ-nullity distribution ($\kappa \in \Re$) in the study of Riemannian manifolds (M, g), which is defined for any $p \in M$ and $k \in \Re$ as follows

$$N(\kappa): p \to N_p(\kappa) = \{W \in T_pM: R(U, V)W = \kappa[g(V, W)U - g(U, W)V]\}$$

for any $U, V \in T_pM$, where T_pM indicate the tangent vector space of M at any point $p \in M$ and R means the Riemannian curvature tensor of type (1,3). Next Blair, Koufogiorgos and Papantoniou (Blair, et al., 1995) introduced the (κ, μ)-nullity distribution which is a generalized notion of the κ-nullity distribution on a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ and defined for any $p \in M^{2n+1}$ and $\kappa, \mu \in \Re$ as follows

$$N(\kappa, \mu): p \to N_p(\kappa, \mu)$$

$= \{W \in T_pM^{2n+1}: R(U,V)W = \kappa[g(V,W)U - g(U,W)V] + \mu[g(V,W)hU - g(U,W)hV]\}$ (1,1)

where $h = \frac{1}{2}\ell_{\xi}\phi$, and ℓ denotes the Lie differentiation.

In (Dilego & Pastore, 2009), Dileo and Pastore introduced the notion of $(\kappa,\mu)'$-nullity distribution, another generalized notion of the κ-nullity distribution, on an almost Kenmotsu manifold $(M^{2n+1},\phi,\xi,\eta,g)$, which is defined for any

$p \in M^{2n+1}$ and $\kappa,\mu \in \Re$ as given

$$N(\kappa,\mu)': p \to N_p(\kappa,\mu)'$$

$= \{W \in T_pM^{2n+1}: R(U,V)W = \kappa[g(V,W)U - g(U,W)V] + \mu[g(V,W)h'U - g(U,W)h'V]\}$ (1,2)

where $h' = h \circ \phi$.

Kenmotsu (Kenmotsu, 1972) introduced new type of almost contact metric manifolds called Kenmotsu manifolds these days. To take into account M^{2n+1} be an almost contact metric manifold with almost contact structure (ϕ,ξ,η,g) given by a $(1,1)$-tensor field ϕ, a characteristic vector field ξ, a 1-form η and a compatible metric g satisfying the conditions (Blair, 1976; Blair, 2010).

$$\phi^2 = -I + \eta \otimes \xi,\ \phi\xi = 0,\ \eta(\xi) = 1,\ \eta \circ \phi = 0, \tag{1.3}$$

$$g(\phi U,\phi V) = g(U,V) - \eta(U)\eta(V) \tag{1.4}$$

for any vector fields U and V of T_pM^{2n+1}. The fundamental 2-form Ω is defined by $\Omega(U,V) = g(U,\phi V)$. The restriction for an almost contact metric manifold being normal is analogous to vanishing of the $(1,2)$-type torsion tensor N_ϕ, defined by $N_\phi = [\phi,\phi] + 2d\eta \otimes \xi$, where $[\phi,\phi]$ is the Nijenhuis torsion of ϕ (Blair, 1976). A normal almost Kenmotsu manifold is a Kenmotsu manifold such that $d\eta = 0$ and $d\Omega = 2\eta \wedge \Omega$. Also Kenmotsu manifolds can be characterized by $(\nabla_U\phi)(V) = g(\phi U,V)\xi - \eta(V)\phi U$, for any vector fields U,V. It is well known (Kenmotsu, 1972) that a Kenmotsu manifold M^{2n+1} is locally a warped product $\hat{I} \times_f N^{2n}$, where N^{2n} is a K$\ddot{a}$hler manifold, $\hat{I}$ is an open interval with coordinate t and the warping function $\hat{f}$, defined by $\hat{f} = ce^t$ for some positive constant c. Let us denote the distribution orthogonal to ξ by $\widehat{D}$ and defined by $\widehat{D} = Ker(\hat{\eta}) = \hat{I}_m(\phi)$. In an almost Kenmotsu manifold, since $\hat{\eta}$ is closed, $\widehat{D}$ is an integrable distribution.

2. ALMOST KENMOTSU MANIFOLDS

Let M^{2n+1} be an almost Kenmotsu manifold. We denote by $h = \frac{1}{2}\ell_\xi\phi$ and $l = R(\cdot,\xi)\xi$ on M^{2n+1}. The tensor fields l and h are symmetric operators and satisfy the blowing in the same direction

$$\xi = 0,\ \ l\xi = 0,\ \ tr(h) = 0,\ \ tr(h\phi) = 0,\ \ h\phi + \phi h = 0. \tag{2.1}$$

Furthermore, for the subsequent results (Dileo, 2007; Gray, 1966).

$$\nabla_U\xi = -\phi^2 U (\Rightarrow \nabla_\xi\xi = 0) \tag{2.2}$$

$$\phi l\phi - l = 2(h^2 - \phi^2) \tag{2.3}$$

$$R(U,V)\xi = \eta(U)(V - \phi hV) - \eta(V)(U - \phi hU) + (\nabla_V\phi h)V - (\nabla_U\phi h)V \tag{2.4}$$

for any vector fields U, V. The (1,1)-type symmetric tensor field $h' = h \circ \phi$ is anticommuting with ϕ and

$$h'\xi = 0$$

Also it is explicit that

$$h = 0 \Leftrightarrow h' = 0,\ \ h'^2 = (\kappa + 1)\phi^2\ \ (\Leftrightarrow h^2 = (\kappa + 1)\phi^2) \tag{2.5}$$

which grip on $(\kappa,\mu)'$-almost Kenmotsu manifold.

In 2014, Shaikh and Kundu (Shaikh & Kundu, 2014) to imported and studied a type of tensor field, called generalized B curvature tensor on a Riemannian manifold. It count the structures of quasi-conformal, Weyl-conformal, conharmonic and concircular curvature tensors and it spell out as

$$B(U,V)W = p_0 R(U,V)W + p_1[S(V,W)U - S(U,W)V + g(V,W)QU - g(U,W)QV]$$

$$+2p_{2r}[g(V,W)U - g(U,W)V] \tag{2.6}$$

where $R, S, \hat{Q}$ and r are the curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature respectively.

Thus, the B-curvature tensor is classified as

i) The quasi-conformal curvature tensor C (Yano & Sawaki, 1968) if

$$p_0 = a, p_1 = b \text{ and } p_2 = -\frac{1}{2n}[\frac{a}{n-1} + 2b] \tag{2.7}$$

ii) The weyl-conformal curvature tensor $\overline{C}$ (Yano, 1984) if

$$p_0 = 1, p_1 = -\frac{1}{n-2} \text{ and } p_2 = -\frac{1}{2(n-1)(n-2)} \tag{2.8}$$

iii) The concircular curvature tensor C^*(Yano, 1940) if

$$p_0 = 1, p_1 = 0 \text{ and } p_2 = -\frac{1}{n(n-1)} \tag{2.9}$$

iv) The conharmonic curvature tensor H (Ishi, 1957) if

$$p_0 = 1, p_1 = -\frac{1}{(n-1)} \text{ and } p_2 = 0 \tag{2.10}$$

3. ξ LINKED WITH (κ,μ)-NULLITY DISTRIBUTION

In this section we consider almost Kenmotsu manifolds with ξ belonging to the (κ,μ)-nullity distribution satisfying B=0, ξ-B flat and ϕ-B flat, where B is the generalized B-curvature tensor. Then from (1.3) we have

$$R(U,V)\xi = \kappa[\eta(V)U - \eta(U)V] + \mu[\eta(V)hU - \eta(U)hV] \tag{3.1}$$

where $\kappa, \mu \in \mathfrak{R}$. Before cited our main theorem, we recall some results.

Theorem 3.1 (Dileo & Pastore, 2009). Let M^{2n+1} be an almost Kenmotsu manifold of dimension $(2n+1)$. Such that the characteristic vector field ξ linked to the (κ,μ)-nullity distribution. Then $\kappa = -1$, $h = 0$ and M^{2n+1} is locally a warped product of an open interval and an almost Kähler manifold.

With the hand of (3.1) and Theorem 3.1 we have the following properties

$$R(U,V)\xi = \eta(U)V - \eta(V)U \tag{3.2}$$

$$\eta(R(U,V)W) = g(U,W)\eta(V) - g(V,W)\eta(U) \tag{3.3}$$

$$R(\xi,U)V = -g(U,V)\xi + \eta(V)U \tag{3.4}$$

$$S(\xi,U) = -2n\eta(U) \tag{3.5}$$

$$\hat{Q}\xi = -2n\xi \tag{3.6}$$

for any vector fields U,V on M^{2n+1}. Therefore, we prove the following result.

Theorem 3.2 An almost Kenmotsu manifold M^{2n+1} with ξ belongs to the (κ,μ)-nullity distribution is B-flat if and only if the manifold is locally isometric to the hyperbolic space $H^{2n+1}(-1)$.

Proof. Let M^{2n+1} is B-flat, that is, $B(U,V)W = 0$, for any vector fields U,V,W on M^{2n+1}. So from (2.6), we have

$$\hat{R}(U,V,W,P) = -\frac{p_1}{p_0}[S(V,W)g(U,P) - S(U,W)g(V,P) + g(V,W)S(U,P) - g(U,W)S(V,P)]$$

$$-\frac{2p_2 r}{p_0}[g(V,W)g(U,P) - g(U,W)g(V,P)] \tag{3.7}$$

On substituting $U = P = e_i,\ 1 \le i \le 2n+1$ in (3.7), where e_i is an orthonormal basis for the tangent space at each point of the manifold. Then

$$S(V,W) = \gamma_1 g(V,W) + \gamma_2 \eta(V)\eta(W) \tag{3.8}$$

where $\gamma_1 = \frac{1}{p_1}[p_0 - 2rp_2 - 2np_1]$ and $\gamma_2 = \frac{1}{p_1}[2p_2 r - p_0]$.

Also, $\gamma_1 + \gamma_2 = -2n$, keeping in mind this fact we obtain from (3.8) that

$$r = 2n(\gamma_1 - 1) \tag{3.9}$$

In (Dileo & Pastore, 2009), Dileo and Pastore prove that in an almost Kenmotsu manifold with ξ belonging to the (κ,μ)-nullity distribution the sectional curvature $L(U,\xi)$=-1. Due to this an almost Kenmotsu manifold with ξ belonging to the (κ,μ)-nullity distribution the scalar curvature $r=-2n(2n+1)$. With the comfort of this value of r get from (3.9) that γ_1=-$2n$ and γ_2=0. So (3.8) reduces to

$$S(V,W)=-2ng(V,W) \tag{3.10}$$

In outlook of (3.9) and (3.10), Eq.(3.7) take the form

$$R(U,V)W=-[g(V,W)U-g(U,W)V] \tag{3.11}$$

That is, the manifold is locally isometric to the hyperbolic space $H^{2n+1}(-1)$. Conversely, if (3.11) holds on M^{2n+1}. On contacting (3.11) gives $S(V,W)=-2ng(V,W)$. Therefore from (2.6) (3.10) and (3.11), we get $B(U,V)W=0$. Thus the theorem is proved. ■

Corollary 3.3 A B-flat almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (κ, μ)-nullity distribution is an Einstein manifold provided B-curvature tensor is not the concircular curvature tensor.

In particular, if $p_0=a,\ p_1=b,\ p_2=\frac{-1}{2n}[\frac{a}{n-1}+2b]$ then B-curvature tensor reduces to the quasi-conformal curvature tensor. Hence we can attitude the following

Corollary 3.4 An almost Kenmotsu manifold M^{2n+1} along ξ belonging to the (κ,μ)-nullity distribution is quasi-conformally flat if and only if the manifold is locally isometric to the hyperbolic space $H^{2n+1}(-1)$.

This result also has been proved by De and Majhi (De & Majhi, 2018).

Corollary 3.5 An almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (κ,μ)-nullity distribution is weyl-conformally flat if and only if the manifold is locally isometric to the hyperbolic space $H^{2n+1}(-1)$.

Corollary 3.6 An almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (κ,μ)-nullity distribution is conharmonically flat if and only if the manifold is locally isometric to the hyperbolic space $H^{2n+1}(-1)$.

Theorem 3.7 An almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (κ,μ)-nullity distribution is ξ-B flat if and only if the manifold is an Einstein manifold provided B-curvature tensor is not the concircular curvature tensor.

Proof. Let the manifold M^{2n+1} is ξ-B flat, that is $B(U,V)\xi = 0$. Thus from (2.6), we have

$$R(U,V)\xi = -\frac{p_1}{p_0}[S(V,\xi)U - S(U,\xi)V + g(V,\xi)QU - g(U,\xi)QV] - \frac{2p_2 r}{p_0}[g(V,\xi)U - g(U,\xi)V] \qquad (3.12)$$

Using (3.2), (3.5) and (3.6) in (3.12) and taking inner product with U, we get

$$[g(V,P)\eta(U) - g(U,P)\eta(V)] = -\frac{p_1}{p_0}[-2n\eta(V)g(U,P) + 2n\eta(U)g(V,P) + \eta(V)S(U,P) - \eta(U)S(V,P)] - \frac{2p_2 r}{p_0}[\eta(V)g(U,P) - \eta(U)g(V,P)]. \qquad (3.13)$$

Putting $U = \xi$ in (3.13), it yields

$$S(V,P) = \gamma_1 g(V,P) + \gamma_2 \eta(V)\eta(P) \qquad (3.14)$$

where $\gamma_1 = \frac{1}{p_1}[p_0 - 2rp_2 - 2np_1]$ and $\gamma_2 = \frac{1}{p_1}[2p_2 r - p_0]$. Also we have $\gamma_1 + \gamma_2 = -2n$, using this fact we have from (3.14) that

$$r = 2n(\gamma_1 - 1). \qquad (3.15)$$

Where as in an almost Kenmotsu manifold with ξ belonging to the (κ,μ)-nullity distribution the scalar curvature $r = -2n(2n+1)$. With the hand of this rate of r get from (3.15) that $\gamma_1 = -2n$ and $\gamma_2 = 0$. So (3.14) reduces to

$$S(V,P) = -2ng(V,P). \qquad (3.16)$$

That is, the manifold is an Einstein. Its converse statement is obvious. This outright the proves. ■

Corollary 3.8 An almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (κ, μ)-nullity distribution is ξ-quasi-conformally flat if and only if the manifold is an Einstein manifold.

This result also proved by De and Majhi (De & Majhi, 2018).

Corollary 3.9 An almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (κ, μ)-nullity distribution is ξ-weyl-conformally flat if and only if the manifold is an Einstein manifold.

Corollary 3.10 An almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (κ, μ)-nullity distribution is ξ-conharmonically flat if and only if the manifold is an Einstein manifold.

Theorem 3.11 An almost Kenmotsu manifold M^{2n+1} with ξbelonging to the (κ, μ)-nullity distribution is ϕ-B flat then the manifold is an Einstein manifold provided the B-curvature tensor is not the concircular curvature tensor.

Proof. Let the manifold M^{2n+1} is ϕ-B flat, that is, $\phi^2(B(\phi U, \phi V)\phi W) = 0$. Then from (2.6) and (3.3) after that pandemic the inner product with P, we have

$$\begin{aligned} 0 = p_0[g(\phi V, \phi W)g(\phi U, P) - g(\phi U, \phi W)g(\phi V, P)] \\ +p_1[2ng(\phi V, \phi W)g(\phi U, P) - 2ng(\phi U, \phi W)g(\phi V, P) \\ +g(\phi V, \phi W)S(\phi U, P) + g(\phi X, \phi W)S(\phi V, P)] \\ -2rp_2[g(\phi V, \phi W)g(\phi U, P) - g(\phi U, \phi)g(\phi V, P)] \end{aligned} \tag{3.17}$$

Taking $V = W = e_i$, $1 \le i \le 2n + 1$ in (3.17), where e_i is an orthonormal basis for the tangent space at each point of the manifold. Then

$$S(\phi U, P) = \left\{\frac{n(p_0 + 2np_1 - 2rp_2)}{p_1(n+1)}\right\} g(\phi U, P). \tag{3.18}$$

Replacing P by ϕP in (3.18) and proving (1.5) and (3.2), we annex

$$S(U, P) = \gamma_1 g(U, P) + \gamma_2 \eta(U)\eta(P), \tag{3.19}$$

where $\gamma_1 = \frac{n(p_0+2np_1-2rp_2)}{p_1(n+1)}$ and $\gamma_2 = -\frac{n(p_0+2p_1-2rp_2+4np_1)}{p_1(n+1)}$.

Also we notice that $\gamma_1 + \gamma_2$=-$2n$, using this evidence we have from (3.19) that

$$r = 2n(\gamma_1 - 1). \tag{3.20}$$

As well an almost Kenmotsu manifold with ξ belonging to the (κ, μ)-nullity distribution the scalar curvature $r = -2n(2n + 1)$. With the hand of this value of r get from (3.20) that $\gamma_1 = -2n$ and $\gamma_2 = 0$. So (3.19) reduces to

$$S(U, P) = -2ng(U, P) \tag{3.21}$$

This outright the proves. ■

Corollary 3.12 An almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (κ, μ)-nullity distribution is ϕ-quasi-conformally flat if and only if the manifold is an Einstein manifold provided the B-curvature tensor is not the concircular curvature tensor.

Corollary 3.13 An almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (κ, μ)-nullity distribution is ϕ-weyl-conformally flat if and only if the manifold is an Einstein manifold provided the B-curvature tensor is not the concircular curvature tensor.

Corollary 3.14 An almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (κ, μ)-nullity distribution is ϕ-conharmonically flat if and only if the manifold is an Einstein manifold provided the B-curvature tensor is not the concircular curvature tensor.

4. SEMISYMMETRIC AND RECURRENT PROPERTIES

Here we consider certain curvature properties, that is, $R \cdot B = 0$, $B \cdot \phi = 0$ and B-ϕ-recurrent on an almost Kenmotsu manifold with ξ belonging to the (κ, μ)-nullity distribution. With the hand of the above properties first we prove that

Theorem 4.1 A B-semisymmetric an almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (κ, μ)-nullity distribution is an Einstein manifold provided the B-curvature tensor is not the concircular curvature tensor.

Proof. Let the manifold under discussion is B-semisymmetric, that is $R \cdot B = 0$. Thus it ensure that

$$0 = R(\xi, P)B(U,V)W - B(R(\xi, P)U, V)W - B(U, R(\xi, P)V)W - B(U,V)R(\xi, P)W \tag{4.1}$$

An exploit of (3.4) in (4.1), we yield

$$0 = \eta(B(U,V)W)P - g(P, B(U,V)W)\xi - \eta(U)B(P,V)W + g(P,U)B(\xi,V)W$$

$$-\eta(V)B(U,P)W + g(P,V)B(U,\xi)W) - \eta(W)B(U,V)P + g(P,W)B(U,V)\xi. \tag{4.2}$$

Using (2.6) and taking the inner product of (4.2) with respect to ξ, we have

$$0 = p_0\eta(R(U,V)W)\eta(P) + p_1[S(V,W)\eta(U)\eta(P) - S(U,W)\eta(V)\eta(P)$$

$$-2ng(V,W)\eta(U)\eta(P) + 2ng(U,W)\eta(V)\eta(P)] + 2rp_2[g(V,W)\eta(U)\eta(P)$$

$$-g(U,W)\eta(V)\eta(P)] - g(P, B(U,V)W) - \eta(U)\eta(B(P,V)W) + g(P,U)\eta(B(\xi,V)W)$$

$$-\eta(V)\eta(B(U,P)W) + g(P,V)\eta(B(U,\xi)W) - \eta(W)\eta(B(U,V)P) + g(P,W)\eta(B(U,V)\xi) \tag{4.3}$$

In the hand of (2.6),(3.2) and (3.5) in (4.3) and then contracting gives

$$S(V,W) = \gamma_1 g(V,W) + \gamma_2\eta(V)\eta(W) \tag{4.4}$$

where $\gamma_1 = \frac{r[(2(2n-1)p_2+1)]}{-[p_0+(2n-1)p_1]}$ and $\gamma_2 = \frac{[p_0-4nrp_2(p_1-r)-8n^2p_1]}{-[p_0+(2n-1)p_1]}$. This completes the proof. ■

Theorem 4.2 A B-ϕ-semisymmetric an almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (κ,μ)-nullity distribution is an η-Einstein manifold provided the B-curvature tensor is not the concircular curvature tensor.

Proof. Let the manifold M^{2n+1} is B-ϕ-semisymmetric, that is $B \cdot \phi = 0$. Then this implies

$$B(U,V)\phi W - \phi B(U,V)W = 0 \tag{4.5}$$

With the hand of (2.6) and (3.3), we have

$$B(U,V)\phi W = p_0[g(U,\phi W)V - g(V,\phi W)U] + p_1[S(V,\phi W)U - S(U,\phi W)V$$

$$+g(V,\phi W)QU - g(U,\phi W)QV] + 2p_2r[g(V,\phi W)U - g(U,\phi W)V] \quad (4.6)$$

and

$$\phi B(U,V)W = p_0[g(U,W)\phi V - g(V,W)\phi U] + p_1[S(V,W)\phi U - S(U,W)\phi V$$

$$+g(V,W)\phi QU - g(U,W)\phi QV] + 2p_2r[g(V,W)\phi U - g(U,W)\phi V]. \quad (4.7)$$

Using (4.6) and (4.7) in (4.5) and after taking the inner product with P, we get

$$0 = p_0[g(U,\phi W)g(V,P) - g(V,\phi W)g(U,P) - g(U,W)g(\phi V,P) + g(V,W)g(\phi U,P)]$$

$$+p_1[S(V,\phi W)g(U,P) - S(U,\phi W)g(V,P) + g(V,\phi W)S(U,P)$$

$$-g(U,\phi W)S(V,P) - S(V,W)g(\phi U,P) + S(U,Z)g(\phi V,P) - g(V,W)S(\phi U,P) + g(U,W)S(\phi V,P)]$$

$$+2p_2r[g(V,\phi W)g(U,P) - g(U,\phi W)g(V,P) - g(V,W)g(\phi U,P) + g(U,W)g(\phi V,P)]. \quad (4.8)$$

Taking $V = P = e_i$, $1 \le i \le 2n+1$ in (4.8), where e_i is an orthonormal basis for the tangent space at each point of the manifold. Then

$$2p_1[S(U,\phi W) - S(\phi U,W)] + [2np_0 - 2(2n-1)rp_2 + (2n+1) - r]g(U,\phi W) = 0. \quad (4.9)$$

Setting U by ϕU in (4.9) and using(1.5), (3.5), we obtain

$$S(U,W) = \gamma_1 g(U,W) + \gamma_2\eta(U)\eta(W) \quad (4.10)$$

where $\gamma_1 = -\frac{\alpha}{4p_1}$, $\gamma_2 = \frac{\alpha - 8np_1}{4p_1}$ and $\alpha = [2np_0 - 2rp_2(2n-1) - p_1(r-(2n+1))]$. Hence the theorem is attest next, we show the following result . ■

Theorem 4.3 A B-ϕ-recurrent an almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (κ,μ)-nullity distribution with constant scalar curvature is a B-ϕ-symmetric manifold provided B-curvature tensor is not conharmonic curvature tensor.

Proof. Let M^{2n+1} under consideration is a B-ϕ-recurrent manifold then there exists a non-zero 1-form ψ such that

$$\phi^2((\nabla_P B)(U,V)W) = \psi(P)B(U,V)W \tag{4.11}$$

for any vector fields $U,V,W,P \in T_p(M)$. If $\psi(P)$=0 then B-ϕ-recurrent manifold reduces to the B-ϕ-symmetric manifold. Then in view of (1.4) and (4.11), we have

$$-(\nabla_P B)(U,V)W) + \eta((\nabla_P B)(U,V)W)\xi = \psi(P)B(U,V)W \tag{4.12}$$

Equation (4.12) can be reduces

$$-g((\nabla_P B)(U,V)W),Q) + \eta((\nabla_P B)(U,V)W)\eta(Q) = \psi(P)g(B(U,V)W,Q) \tag{4.13}$$

With the hand of (2.6),(3.2) and (3.5), Eq.(4.13) cut down

$$-p_0 g((\nabla_P R)(U,V)W,Q) - p_1[g((\nabla_P S)(V,W)U,Q) - g((\nabla_P S)(U,W)V,Q)$$

$$+g(V,W)(\nabla_P S)(U,Q) - g(U,W)(\nabla_P S)(V,Q)] - 2p_2 dr(P)[g(V,W)g(U,Q) - g(U,W)g(V,Q)]$$

$$+\eta(Q)p_0 g((\nabla_P R)(U,V)W,\xi) + p_1[g((\nabla_P S)(V,W)U,\xi)$$

$$-g((\nabla_P S)(U,W)V,\xi) + g(V,W)(\nabla_P S)(U,\xi) - g(U,W)(\nabla_P S)(V,\xi)]$$

$$+2p_2 dr(P)[g(V,W)g(U,\xi) - g(U,W)g(V,\xi)]$$

$$= \psi(P)p_0 g(R(U,V)W,Q) + p_1[S(V,W)g(U,Q) - S(U,W)g(V,Q)$$

$$+g(V,W)S(U,Q) - g(U,W)S(V,Q)] + 2p_2 r[g(V,W)g(U,Q) - g(U,W)g(V,Q)]. \tag{4.14}$$

Contracting (4.14) and after simplification we get

$$2(p_0 - p_1)(\nabla_P S)(V,W) + 4np_2 dr(P)g(V,W)$$

$$+\eta(W)g(V,Q)-\eta(V)\eta(W)\eta(W)+2p_2dr(P)[\eta(U)g(V,W)-\eta(V)g(U,W)]$$

$$=\psi(P)[(p_0-3p_1-2np_1)S(V,W)+(p_1+4np_2)rg(V,W)]. \tag{4.15}$$

Substituting $V=W=\xi$ in (4.15), we have

$$\psi(P)=\frac{4n(2n+1)p_2dr(Q)}{[-2n(p_0-(2n+3))p_1+(2n+1)r(p_1+4np_2)]}. \tag{4.16}$$

If the manifold under consideration is of constant curvature, then $dr(Q)=0$. Consequently, we get $\psi(P)=0$. Therefore from (4.11), we get $\phi^2\big((\nabla_P B)(U,V)W\big)=0$, that is manifold reduces to B-ϕ-symmetric manifold. Hence the theorem is justify. . ■

Corollary 4.4 A conharmonically ϕ-recurrent an almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (κ,μ)-nullity distribution is always ϕ-symmetric manifold.

5. CURVATURE CONDITIONS $B\cdot R=0$, $B\cdot B=0$ AND $B\cdot S=0$

Next, we suppose that the manifold satisfying some condition, that is, $B\cdot R=0$, $B\cdot B=0$ and $B\cdot S=0$, where B, R and S are the B-curvature tensor, the Riemannian curvature tensor and the Ricci tensor respectively. Now, in this position we show the theorem.

Theorem 5.1 An almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (κ,μ)-nullity distribution the condition $B\cdot R=0$, then the manifold is an Einstein manifold provided the B-curvature tensor is not the concircular curvature tensor.

Proof. We consider M^{2n+1} satisfies the condition $B\cdot R=0$. Then

$$0=B(\xi,P)R(U,V)W-R(B(\xi,P)U,V)W-R(U,B(\xi,P)V)W-R(U,V)B(\xi,P) \tag{5.1}$$

In view of (2.6), (3.2) and (3.5), equation (5.1) can be reduces on taking $W = \xi$

$$p_1[S(V,P)U - 4ng(U,P)V] - 4rp_2[g(U,P)V + \eta(U)\eta(P)V] = 0. \quad (5.2)$$

Now taking the inner product of (5.2) with Q, we obtain

$$0 = p_1[S(V,P)g(U,Q) - 4nS(U,P)g(V,Q)] - 4rp_2[g(U,P)g(V,Q) + \eta(U)\eta(P)g(V,Q)]. \quad (5.3)$$

Taking $U = P = e_i$, $1 \leq i \leq 2n+1$ in (5.3), where e_i is an orthonormal basis for the tangent space at each point of the manifold, we have

$$S(V,Q) = 4r\left\{n + \frac{2p_2}{p_1}(n+1)\right\}g(V,Q). \quad (5.4)$$

This achieve the proof. . ■

Theorem 5.2 An almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (κ,μ)-nullity distribution satisfying the condition $B \cdot B = 0$, then the manifold is provided the B-curvature tensor is not the concircular curvature tensor.

Proof. Let M^{2n+1} satisfying the condition $B \cdot B = 0$, which implies

$$0 = B(\xi,P)B(U,V)\xi - B(B(\xi,P)U,V)\xi - B(U,B(\xi,P)V)\xi - B(U,V)B(\xi,P)\xi. \quad (5.5)$$

Applying (2.6), (3.2), (3.5) in (5.5) and then taking the inner product with ξ and at $U = \xi$, we obtain

$$(p_0 + 2np_1 - 2rp_2)[\hat{Q}P - \gamma_1 P - \gamma_2\eta(P)\xi] = 0 \quad (5.6)$$

which implies that either $p_0 = 2(rp_2 - np_1)$ or $\hat{Q}P = \gamma_1 P + \gamma_2\eta(P)\xi$, where $\gamma_1 = \frac{1}{p_1}[2np_1 + 4p_2(r+1)]$, $\gamma_2 = -\frac{1}{p_1}[p_0 + 6np_1 + 2rp_2]$. ■

Finally, we show that

Theorem 5.3 An almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (κ,μ)-nullity distribution the curvature condition $B \cdot S = 0$, if and only if the manifold is an Einstein manifold.

Proof. Let the condition $B \cdot S = 0$ holds on M^{2n+1} which implies that $(B(U,V) \cdot S)(W,P) = 0$, for all vector fields U, V, W, P. Then we have

$$S(B(U,V)W,P) + S(W,B(U,V)P) = 0 \tag{5.7}$$

for any vector fields U, V, W, P on M^{2n+1}. Substituting U=W=ξ in (5.7) we have

$$S(B(\xi,V)\xi,P) + S(\xi,B(\xi,V)P) = 0 \tag{5.8}$$

By the use of (2.6), (3.4), (3.5) and (3.6) we get from (5.8) that

$$S(V,P) = \gamma_1 g(V,P) \tag{5.9}$$

where $\gamma_1 = \frac{4nrp_2 - 4n^2p_1 - 2np_0}{p_0 + 2(np_1 - rp_2)}$. Thus the manifold is an Einstein manifold. Conversely, if the manifold under consideration is an Einstein manifold, then from (5.7) it follows that $B \cdot S = 0$ holds identically. This execute the proof of the theorem. ■

6. ξ LINKED WITH $(\kappa,\mu)'$-NULLITY DISTRIBUTION

This section is related to if $U \in \widehat{D}$ be the eigen vector of h' corresponding to the eigen value λ. Then from (2.5) it is light that $\lambda^2 = -(k+1)$, a constant. Hence $k \leq 1$ and $\lambda = \pm\sqrt{-k+1}$. We indicate the eigen spaces associated with h' by $[\lambda]'$ and $[-\lambda]'$ corresponding to the non-zero eigen values $[\lambda]$ and $[-\lambda]$ of h' respectively. Thus we recall some results.

Lemma 6.1 (Wang & Lui, 2015). Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belonging to the $(\kappa,\mu)'$-nullity distribution. If $h' \neq 0$, then the Ricci operator $\hat{Q}$ of M^{2n+1} is given by

$$\hat{Q} = -2nid + 2n(k+1)\eta \otimes \xi - 2nh' \tag{6.1}$$

In addition, the scalar curvature of M^{2n+1} is $2n(k-2n)$.

Also in an almost Kenmotsu manifold with ξ belonging to the $(\kappa,\mu)'$-nullity distribution, we carry

$$R(U,V)\xi = k[\eta(V)U - \eta(U)V] + \mu[\eta(V)h'U - \eta(U)h'V] \tag{6.2}$$

$$R(\xi,U)V = k[g(U,V)\xi - \eta(V)U] + \mu[g(h'U,V)\xi - \eta(V)h'U] \tag{6.3}$$

$$S(V,\xi) = 2nk\eta(V). \tag{6.4}$$

Now we show the following result.

Theorem 6.2 A (2n + 1)-dimensional $(n > 1)$ B-flat almost Kenmotsu manifold with ξ belonging to the $(\kappa,\mu)'$-nullity distribution is either conharmonically flat or of a quasi-constant curvature.

Proof. $\hat{R}(U,V,W,P) = -\frac{p_1}{p_0}[S(V,W)g(U,P) - S(U,W)g(V,P) + g(V,W)S(U,P) - g(U,W)S(V,P)]$

$$-\frac{2p_2r}{p_0}[g(V,W)g(U,P) - g(U,W)g(V,P)]. \tag{6.5}$$

Taking $V = W = \xi$ in (6.5), using (6.2) and (6.4) we get after simplifying

$$S(U,P) = \gamma_1 g(U,P) + \gamma_2\eta(U)\eta(V) - \frac{p_0\mu}{p_1}g(h'U,P) \tag{6.6}$$

where $\gamma_1 = -[\frac{p_0k}{p_1} + \frac{2rp_2}{p_1} + 2nk]$, and $\gamma_2 = [\frac{p_0k}{p_1} + \frac{2rp_2}{p_1} + 4nk]$. It is noticed that

$$\gamma_1 + \gamma_2 = 2nk \tag{6.7}$$

With the hand of (6.6) and (6.7), we obtain

$$\gamma_1 = \frac{r}{2n} - k \tag{6.8}$$

This complete the prove. ■

7. EXTENDED *B*-CURVATURE TENSOR OF AN ALMOST KENMOTSU MANIFOLD WİTH (κ,μ)- NULLITY DISTRIBUTION

This section concern with the light of vanishing extended B-curvature tensor and extended ξ-B-flat almost Kenmotsu manifolds with ξ belonging to the

(κ,μ)-nullity distribution. The extended form of generalized B-curvature tensor can be designate as

$$B_e(U,V)W = p_0R(U,V)W + p_1[S(V,W)U - S(U,W)V + g(V,W)QU - g(U,W)QV]$$

$$+2p_2r[g(V,W)U - g(U,W)V] - \eta(U)B(\xi,V)W - \eta(V)B(U,\xi)W - \eta(W)B(U,V)\xi \qquad (7.1)$$

Now we came to the following result.

Theorem 7.1 In an almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (κ,μ)-nullity distribution, if the extended B-curvature tensor vanishes then the manifold is η-Einstein provide the B-curvature tensor is not the concircular curvature tensor.

Proof. Let $B_e(U,V)W = 0$ holds on M^{2n+1}. So adopting $V = W = \xi$, we get from (7.1) that

$$R(U,\xi)\xi = -\frac{p_1}{p_0}[S(\xi,\xi)U - S(U,\xi)\xi + g(\xi,\xi)QU - g(U,\xi)Q\xi]$$

$$-\frac{2rp_2}{p_0}[g(\xi,\xi)U - g(U,\xi)\xi] + \frac{1}{p_0}[\eta(U)B(\xi,\xi)\xi + \eta(\xi)B(U,\xi)\xi + \eta(\xi)B(U,\xi)\xi] \qquad (7.2)$$

Now, making the use of (3.2), (3.5) and (3.6) the equation (7.2) reflect as

$$2B(U,\xi)\xi = -[p_0 + 2np_1 - 2rp_2]U + [p_0 + 4np_1 - 2rp_2]\eta(U)\xi + p_1QU \qquad (7.3)$$

On the other-hand in view of (2.6),(3.2),(3.5) and (3.6) we obtain

$$(U,\xi)\xi = -[p_2 + 2np_1 - 2rp_2]U + [p_0 + 4np_1 - 2rp_2]\eta(U)\xi + p_1QU. \quad (7.4)$$

With the help of (7.3) and (7.4), we get

$$\hat{Q}U = \gamma_1 U + \gamma_2\eta(U)\xi \qquad (7.5)$$

where $\gamma_1 = \frac{1}{p_1}[2np_1 + 2p_2(r+1)]$, and $\gamma_2 = -\frac{1}{p_1}[p_0 + 4np_1 + 2rp_2]$. This shows that the manifold is η-Einstein. Hence the theorem is proved. ■

At last we show the subsequent result.

Theorem 7.2 An almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (κ,μ)-nullity distribution is extended ξ-B flat then the manifold is η-Einstein provide the B-curvature tensor is not the concircular curvature tensor.

Proof. Let the condition $B_e(X,Y)\xi = 0$ holds on M^{2n+1}. So from (7.1), we have

$$R(U,V)\xi = -\frac{p_1}{p_0}[S(V,\xi)U - S(U,\xi)V + g(V,\xi)QU - g(U,\xi)QV] - \frac{2rp_2}{p_0}[g(V,\xi)U - g(U,\xi)V]$$

$$+\frac{1}{p_0}[\eta(U)B(\xi,V)\xi + \eta(V)B(U,\xi)\xi + \eta(\xi)B(U,V)\xi]. \tag{7.6}$$

Taking $V = \xi$ in (7.6) and using (3.2),(3.5) and (3.6) then (7.6) reduces to

$$2B(U,\xi)\xi = [-p_0 - 2np_1 + 2rp_2]U + [p_0 + 4np_1 - 2rp_2]\eta(U)\xi + p_1 QU. \tag{7.7}$$

On the other-hand in view of (2.6),(3.2),(3.5) and (3.6) we obtain

$$B(U,\xi)\xi = -[p_2 + 2np_1 - 2rp_2]U + [p_0 + 4np_1 - 2rp_2]\eta(U)\xi + p_1 QU. \tag{7.8}$$

With the help of (7.7) and (7.8), we get

$$\hat{Q}U = \gamma_1 U + \gamma_2 \eta(U)\xi \tag{7.9}$$

where $\gamma_1 = \frac{1}{p_1}[2np_1 + 2p_2(r+1)]$, and $\gamma_2 = -\frac{1}{p_1}[p_0 + 4np_1 + 2rp_2]$.This implies that the manifold is an η-Einstein. Hence the theorem is established. ■

8. CONCLUSION

In 2014, Shaikh and Kundu (Shaikh & Kundu, 2014) to imported and studied a type of tensor field, called generalized B curvature tensor on a Riemannian manifold. It counts the structures of quasi-conformal, Weyl-conformal, conharmonic and concircular curvature tensors. In this consequences we study certain courvature condition on such curvature on almost Kenmotsu manifolds with its characteristic vector field ξ belongs to the (k,μ)-nullity and $(k,\mu)'$-nullity distribution respectively. The object of the paper is to study almost Kenmotsu manifolds with its characteristic vector field ξ belongs to the (k,μ)-nullity and $(k,\mu)'$-nullity distribution respectively. Also we deal with conditions $B.R$, $B.B$ and $B.S$ in an almost Kenmotsu manifold. As a consequence of the main results we obtain some corollaries.

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